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On weakly f-clean rings

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Abstract. Let R be an associative ring with identity and $Id(R)$ and $K(R)$ denote the set of idempotents and full elements of R respectively. The notion of weakly f-clean rings where element r can be written as $r = f + e$ or $r = f - e$, $e \in Id(R)$ and $f \in K(R)$ was introduced. Different properties of weakly f-clean rings were studied. It was shown that a left quasi-duo ring R is weakly clean if and only if R is a weakly f-clean ring. Finally, it was shown that the ring of skew Hurwitz series $T = (HR, \alpha)$ where α is an automorphism of R is weakly f-clean if and only if R is weakly f-clean.

1. Introduction

Let R be an associative ring with identity and $U(R)$ and $Id(R)$ denote the set of units and idempotents of R respectively. The ring R is clean if for each $r \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $r = u + e$ [2, 15]. A ring R is weakly clean if each $r \in R$ can be written in the form $r = u+e$ or $r = u-e$ where $u \in U(R)$ and $e \in Id(R)$ [1, 5, 8, 13]. Other generalizations of clean rings have been introduced [3, 6, 9, 10, 16] An element $f \in R$ is full element if there exist $x, y \in R$ such that $xfy = 1$. $K(R)$ will denote the set of full elements of R. An element $r \in R$ is said to be f-clean if it can be written as the sum of an idempotent and a full element. A ring R is said to be f-clean if each element in R is a f-clean element [12, 14].

In this paper, we introduce the notion of a weakly f-clean ring as a new generalization of a weakly clean ring and a f-clean ring. Let R be a ring. An element $r \in R$ is called weakly f-clean if there exist $f \in K(R)$ and $e \in Id(R)$ of R such that $r = f + e$ or $r = f - e$. A ring R is called weakly f-clean if every element of R is weakly f-clean. Various properties of weakly f-clean rings and weakly f-clean elements were studied. We showed that, every homomorphic image of a weakly f-clean ring is weakly f-clean and

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$\prod_{i\in I} R_i$ is weakly f-clean if and only if every R_i is weakly f-clean (Lemma 2.8). We also showed that, if R is a weakly f-clean ring and $e \in R$ is a central idempotent, then the corner ring eRe is weakly f-clean (Lemma 2.13). A left quasi-duo ring R is weakly clean if and only if R isa weakly f-clean ring (Theorem 2.17). Finally, we showed that the ring of skew Hurwitz series $T = (HR, \alpha)$ where α is an automorphism of R is weakly f-clean if and only if R is weakly f-clean (Theorem 2.23).

2. Main results

We start our work with the following definition.

Definition 2.1. An element $f \in R$ is said to be a *full element* if there exist $x, y \in R$ such that $xfy = 1$. The set of all full elements of a ring R will be denoted by $K(R)$. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$ [14].

Definition 2.2. An element in R is said to be *f-clean* if it can be written as the sum of an idempotent and a full element. A ring R is called a *f-clean ring* if each element in R is a f-clean element [14].

In the following, we define the weakly f-clean rings. Then we study some of the basic properties of weakly f-clean rings. Moreover, we give some necessarily examples.

Definition 2.3. Let R be a ring. Then an element $r \in R$ is called *weakly f-clean* if there exist $f \in K(R)$ and $e \in Id(R)$ of R such that $r = f + e$ or $r = f - e$. A ring R is called *weakly f-clean* if every element of R is weakly f-clean.

Example 2.4. Every clean, weakly clean or f-clean ring is weakly f-clean. Since every purely infinite simple ring is a f-clean ring, and so is weakly f-clean [14]. $(\mathbb{Z}_8, +, .)$ is a weakly f-clean ring, but $(\mathbb{Z}, +, .)$ is not a weakly f-clean ring.

A weakly f-clean ring is not f-clean, in general.

Example 2.5. Let p and q be two distinct odd primes. Then the ring

$$\mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)} = \left\{ \frac{r}{s} \mid r, s \in \mathbb{Z}, s \neq 0, p \nmid s, q \nmid s \right\}$$

is a weakly f-clean ring that is not f-clean.

Proposition 2.6. *Let R be a ring and $r \in R$. Then r is weakly f-clean if and only if $-r$ weakly f-clean.*

Proof. Suppose that r is weakly f-clean. Hence $r = f + e$ or $r = f - e$ for some $f \in K(R)$ and $e \in Id(R)$. Then $-r = -f - e$ or $-r = -f + e$. Since $-f \in K(R)$, $-r$ weakly f-clean. □

Proposition 2.7. *Let R be a ring and every idempotent of R is central. Then $r \in R$ is weakly f-clean if and only if $1-r$ or $1+r$ is f-clean.*

Proof. Suppose r is weakly f-clean. Hence $r = f + e$ or $r = f - e$ for some $f \in K(R)$ and $e \in Id(R)$. Then $1 - r = -f + (1 - e)$ or $1 + r = f + (1 - e)$, and so $1 - r$ or $1 + r$ is f-clean. Conversely, assume that $1 - r$ or $1 + r$ is f-clean. Hence $1 - r = f + e$ or $1 + r = f + e$ for some $f \in K(R)$ and $e \in Id(R)$. Then $r = -f + (1 - e)$ or $r = f - (1 - e)$, thus r is weakly f-clean. □

Lemma 2.8.

(*i*) *Every homomorphic image of a weakly f-clean ring is weakly f-clean.*

(*ii*) *Let $\{R_i\}_{i\in I}$ be a family of rings. Then $\prod_{i\in I} R_i$ is weakly f-clean if and only if every R_i is weakly f-clean.*

Proof. (*i*). Is clear.
(*ii*). Suppose that every R_i is weakly f-clean and $r = (r_i) \in R$. Hence $r_i = f_i + e_i$ or $r_i = f_i - e_i$ for some $f_i \in K(R_i)$ and $e_i \in Id(R_i)$. Then $r = f + e$ such that $f = (f_i) \in K(R)$ and $e = (e_i) \in Id(R)$, and so R is weakly f-clean. The converse follows from (*i*). □

Let I be an ideal of a ring R. We say that *idempotents of R are lifted modulo I* if, for given $r \in R$ with $r - r^2 \in I$, there exists $e \in Id(R)$ such that $e - r \in Id(R)$ [15].

Lemma 2.9. *Let R be a ring such that idempotents are lifted modulo $J(R)$. Then R is weakly f-clean if and only if $R/J(R)$ is weakly f-clean.*

Proof. Suppose that R is weakly f-clean. Hence $R/J(R)$ is weakly f-clean, by Lemma 2.8. Conversely, assume that $R/J(R)$ is weakly f-clean and $r \in R$. Hence $r+J(R) = (f+J(R))+(e+J(R))$ or $r+J(R) = (f+J(R))-(e+J(R))$ with $e^2-e \in J(R)$ and $(x+J(R))(f+J(R))(y+J(R)) = 1+J(R)$ for some $x, y \in R$. Since idempotents can be lifted modulo $J(R)$, e is an idempotent and $r = f + b + e$ or $r = f + b - e$ for some $b \in J(R)$. Since $(x+J(R))(f+J(R))(y+J(R)) = 1+J(R)$, $xfy = 1+z \in 1+J(R) \subseteq U(R)$ for some $z \in J(R)$. Therefore, there exist $x_1, y_1 \in R$ such that $x_1fy_1 = 1$. Hence $x_1(f+b)y_1 = 1+x_1by_1 \in 1+J(R) \subseteq U(R)$. Thus $x_1(f+b)y_1u^{-1} = 1$ for some $u \in U(R)$, and so $f + b \in K(R)$. Then R is weakly f-clean. □

Lemma 2.10. *Let R be a ring. Then R is weakly f-clean if and only if for every $r \in R$ there exist $g \in Id(R)$ and $f \in K(R)$ such that $gr = gf$ and $(g-1)(r-1) = (g-1)f$ or $gr = gf$ and $(g-1)(r-1) = (g-1)f+2(1-g)$.*

Proof. Suppose that R is weakly f-clean and $r \in R$. Hence $r = f + e$ or $r = f - e$ for some $e \in Id(R)$ and $f \in K(R)$. Assume $g = 1-e$. If $r = f+e$, then $gr = g(f + e) = gf$ and $(g - 1)(r - 1) = (g - 1)f$. If $r = f - e$, then $gr = g(f + e) = gf$ and $(g - 1)(r - 1) = (g - 1)f + 2(1 - g)$. Conversely, assume that for every $r \in R$ there exist $g \in Id(R)$ and $f \in K(R)$ such that $gr = gf$ and $(g - 1)(r - 1) = (g - 1)f$. Then $gf - f = gr - g - r + 1$, and so $r = f + (1 - g)$. If for every $r \in R$ there exist $g \in Id(R)$ and $f \in K(R)$ such that $gr = gf$ and $(g - 1)(r - 1) = (g - 1)f + 2(1 - g)$, then $gf - f + 2(1 - g) = gr - g - r + 1$, and so $r = f - (1 - g)$. Therefore R is weakly f-clean. □

Each polynomial ring over a nonzero commutative ring is not weakly clean [1, Theorem 1.9]. If R is commutative ring, then $U(R) = K(R)$, R is weakly clean if and only if R is weakly f-clean. Hence each polynomial ring over a nonzero commutative ring is not weakly f-clean.

Lemma 2.11. *Let R be a ring such that idempotents are lifted modulo $J(R)$ and $R[\alpha] = R + R\alpha + \cdots + R\alpha^n$ with $\alpha^{n+1} = 0$. Then R is weakly f-clean if and only if $R[\alpha]$ is weakly f-clean.*

Proof. Suppose that R is weakly f-clean. Since $J(R[\alpha]) = J(R) + \langle\alpha\rangle$,

$$R[\alpha]/J(R[\alpha]) \cong R/J(R).$$

Then $R[\alpha]/J(R[\alpha])$ is weakly f-clean, by Lemma 2.9. Since idempotents can be lifted modulo $J(R[\alpha])$, $R[\alpha]$ is weakly f-clean, by Lemma 2.9. Conversely, suppose that $R[\alpha]$ is weakly f-clean. Since $R[\alpha]/J(R[\alpha]) \cong R/J(R)$, $R/J(R)$ is weakly f-clean. Since idempotents can be lifted modulo $J(R)$, R is weakly f-clean, by Lemma 2.9. □

Proposition 2.12. *Let R be a ring and $e \in Id(R)$ such that $r \in eRe$ is weakly f-clean in eRe. Then r is weakly f-clean in R.*

Proof. Suppose $r \in eRe$ is weakly f-clean in eRe. Hence $r = f + g$ or $f - g$ for some $g \in Id(eRe)$ and $f \in K(eRe)$, and so there exist $x, y \in eRe$ such that $xfy = e$. If $r = f + g$, then $(x - (1 - e))(f - (1 - e))(y + (1 - e)) = (xfy + (1 - e)) = 1$, and so $f - (1 - e) \in K(R)$. It is clear that $g + (1 - e) \in Id(R)$. Hence $r = (f - (1 - e)) + (g + (1 - e))$. If $r = f - g$, then $(x + (1 - e))(f + (1 - e))(y + (1 - e)) = (xfy + (1 - e)) = 1$, and so $f + (1 - e) \in K(R)$. It is clear that $g + (1 - e) \in Id(R)$. Hence $r = (f + (1 - e)) - (g + (1 - e))$. Therefore r is weakly f-clean in R. □

Lemma 2.13. *Let R be a weakly f-clean ring and $e \in R$ be a central idempotent. Then the corner ring eRe is weakly f-clean.*

Proof. Assume that R is a weakly f-clean ring and $e \in R$ is a central idempotent. Hence eRe is homomorphic image of R. Then eRe is weakly f-clean, by Lemma 2.8. □

Let R be a ring and ${}_RM_R$ be an R-R-bimodule which is a ring possibly without a unity in which $(mn)r = m(nr)$, $(mr)n = m(rn)$ and $(rm)n = r(mn)$ held for all $m, n \in M$ and $r \in R$. The ideal extension of R by M is defined to be the additive abelian group $I(R, M) = R \oplus M$ with multiplication $(r, m)(s, n) = (rs, rn + ms + mn)$.

Lemma 2.14. *Let R be a weakly f-clean and ${}_RM_R$ be an R-R-bimodule such that for any $m \in M$, there exists $n \in M$ such that $m + n + nm = 0$. Then the ideal-extension $I(R, M)$ of R by M is weakly f-clean.*

Proof. Suppose that $(r, m) \in I(R, M)$. Hence $r = f + e$ or $r = f - e$ for some $e \in Id(R)$ and $f \in K(R)$. Then $(r, m) = (f, m) + (e, 0)$ or $(r, m) = (f, m) - (e, 0)$. It is clear that $(e, 0) \in Id(I(R, M))$. Assume that $xfy = 1$. Hence $xmy \in M$, and so there exists $n \in M$ such that $xmy + n + nxmy = 0$. Then $(x, nx)(f, m)(y, 0) = 1$, and so $(f, m) \in K(Id(I(R, M))$. Therefore $Id(I(R, M)$ is weakly f-clean. □

Let R be a ring and σ be a ring endomorphism of R. Then the skew power series ring $R[[x; \sigma]]$ of R is the ring obtained by giving the formal power series ring over R with the new multiplication $xr = \sigma(r)x$ for all $a \in R$. In particular, $R[[x]] = R[[x; 1_R]]$.

Lemma 2.15. *Let R be a ring and σ be a ring endomorphism of R. Then the following statements are equivalent.*

(*i*) *R is a weakly f-clean ring.*

(*ii*) *The formal power series ring* $R[[x]]$ *of* R *is a weakly* f*-clean ring.*

(*iii*) *The skew power series ring* $R[[x;\sigma]]$ *of* R *is a weakly* f*-clean ring.*

Proof. $(ii) \Rightarrow (i)$. Suppose $R[[x]]$ is a weakly f-clean ring. Since R is a homomorphic image of $R[[x]]$, R is weakly f-clean, by Lemma 2.8.
$(iii) \Rightarrow (i)$. Suppose $R[[x;\sigma]]$ is a weakly f-clean ring. Since R is a homomorphic image of $R[[x;\sigma]]$, R is weakly f-clean, by Lemma 2.8.
$(i) \Rightarrow (iii)$. Suppose R is a weakly f-clean ring and $g = r_0 + r_1x + \cdots \in R[[x;\sigma]]$. Then $r_0 = f_0 + e_0$ or $r_0 = f_0 - e_0$ for some $f_0 \in K(R)$ and $e_0 \in Id(R)$. If $r_0 = f_0 + e_0$ and $g' = g - e_0 = f_0 + r_1x + \cdots$ such that $x_0f_0y_0 = 1$ for some $x_0, y_0 \in R$, then $u = (x_0 + \cdots)g'(y_0 + \cdots) = 1 + x_0r_1\sigma(y_0)x + \cdots \in U(R[[x;\sigma]])$. Hence $g' \in K(R[[x;\sigma]])$, and $g = g' + e_0$ with $e_0 \in Id(R[[x;\sigma]])$. If $r_0 = f_0 - e_0$ and $g' = g + e_0 = f_0 + r_1x + \cdots$ such that $x_0f_0y_0 = 1$ for some $x_0, y_0 \in R$, then $u = (x_0 + \cdots)g'(y_0 + \cdots) = 1 + x_0r_1\sigma(y_0)x + \cdots \in U(R[[x;\sigma]])$. Hence $g' \in K(R[[x;\sigma]])$, and $g = g' - e_0$ with $e_0 \in Id(R[[x;\sigma]])$. Therefore $R[[x;\sigma]]$ is weakly f-clean.
$(i) \Rightarrow (ii)$. Suppose R is a weakly f-clean ring. Since $R[[x]] = R[[x;1_R]]$, the proof is similar to $(i) \Longrightarrow (iii)$. □

Theorem 2.16. *Let* R *be a ring and* $r \in R$ *is a weakly* f*-clean element. Then* $B = \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}$ *is a weakly* f*-clean element in* $M_2(R)$ *for every* $s \in R$.

Proof. Suppose $r \in R$ is a weakly f-clean element. Then $r = f + e$ or $r = f - e$ for some $f \in K(R)$ and $e \in Id(R)$. Hence $xfy = 1$ for some $x, y \in R$. If $r = f + e$, then

$$B = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} f & y \\ 0 & -1 \end{pmatrix},$$

such that $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \in Id(M_2(R))$ and $\begin{pmatrix} f & y \\ 0 & -1 \end{pmatrix} \in K(M_2(R))$, by [14, Proposition 2.6]. If $r = f - e$, then

$$B = \begin{pmatrix} f & s \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}.$$

such that $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \in Id(M_2(R))$ and

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so $\begin{pmatrix} f & s \\ 0 & 1 \end{pmatrix} \in K(M_2(R))$. Therefore $B = \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}$ is a weakly f-clean element in $M_2(R)$ for every $s \in R$. □

A ring R is said to be *left quasi-duo*, if every maximal left ideal of R is a two-sided ideal. Commutative rings, local rings, rings in which every nonunit has a power that is central are all belong to this class of rings [17] . A ring R is said to be *Dedekind finite* if $rs = 1$ always implies $sr = 1$ for any $r, s \in R$.

Theorem 2.17. *Let R be a left quasi-duo ring. Then the following statements are equivalent.*

(i) *R is a weakly clean ring.*

(ii) *R is a weakly f-clean ring.*

Proof. $(i) \Rightarrow (ii)$. Is clear.
$(ii) \Rightarrow (i)$. Suppose R is a weakly f-clean ring. Since R is a left quasi-duo ring, $K(R) \subseteq U(R)$, by [14, Theorem 2.9]. Hence R is a weakly clean ring. □

Corollary 2.18. *Let R be a commutative (local or Dedekind finite) ring. Then R is weakly clean if and only if R is weakly f-clean.*

Proof. Since every commutative (local or Dedekind finite) ring is a left quasi-duo ring, the assertion holds, by Theorem 2.17. □

Corollary 2.19. *Let R be a ring in which every nonunit has a power that is central. Then R is weakly clean if and only if R is weakly f-clean.*

Proof. Suppose every nonunit has a power that is central. Hence R is a left quasi-duo ring. Then the assertion holds, by Theorem 2.17. □

Corollary 2.20. *Let R be a ring in which all idempotents are central. Then R is weakly clean if and only if R is weakly f-clean.*

Proof. Since all idempotents are central, R is Dedekind finite. Hence the assertion holds, by Corollary 2.18. □

If G is a group and R is a ring, we denote the group ring over R by RG.

Lemma 2.21. *Let R be a ring such that $2 \in U(R)$. Then R is weakly f-clean if and only if RG is weakly f-clean.*

Proof. Suppose RG is weakly f-clean. Since R is a homomorphic image of RG, R is weakly f-clean, by Lemma 2.8. Conversely, since $2 \in U(R)$, $RG \cong R \times R$, by [11, Proposition 3]. Hence RG is weakly f-clean by Lemma 2.8. □

Suppose that R is an associative ring with unity and $\alpha : R \longrightarrow R$ is an endomorphism such that $\alpha(1)$. The ring (HR, α) of skew Hurwitz series over a ring R is defined as follows: the elements of (HR, α) are functions $f : \mathbb{N} \longrightarrow R$, where $\mathbb{N}$ is the set of integers greater or equal than zero. The operation of addition in (HR, α) is componentwise and the operation of multiplication is defined, for every $f, g \in (HR, \alpha)$, by:

$$fg(n) = \textstyle\sum_{k=0}^{n} \binom{n}{k} f(k)\alpha^k(g(n-k)) \text{ for each } \ n \in \mathbb{N},$$

where $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}$ with $n \geqslant k$ by $n!/k!(n-k)!$. In the case where the endomorphism α is the identity, we denote HR instead of (HR, α). If one identifies a skew formal power series $\sum_{n=0}^{\infty} \in R[[x; \alpha]]$ with the function f such that $f(n) = a_n$, then multiplication in (HR, α) is similar to the usual product of skew formal power series, except that binomial coefficients appear in each term in the product introduced above. It can be easily shown that T is a ring with identity h_1, defined by $h_1(0) = 1$ and $h_1(n) = 0$ for all $n \geqslant 1$. It is clear that R is canonically embedded as a subring of (HR, α) via $r \in R \longmapsto h_r \in (HR, \alpha)$, where $h_r(0) = r$, $h_r(n) = 0$ for every $n \geqslant 1$ [4, 11].

Proposition 2.22. *Let R be a ring. Then $f \in K(T = (HR, \alpha))$ if and only if $f(0) \in K(R)$.*

Proof. [12, Proposition 2.11]. □

Theorem 2.23. *Let R be a ring and α be an automorphism of R. Then $T = (HR, \alpha)$ is weakly f-clean if and only if R is weakly f-clean.*

Proof. Suppose that $W = \{h \in T \mid h(0) = 0\}$, where $T = (HR, \alpha)$ is weakly f-clean. Hence $R \cong T/W$, and so R is a homomorphic image of T. Then R is weakly f-clean, by Lemm 2.8. Conversely, asuume that R is weakly f-clean and $h \in T$. Hence $h(0) \in R$, and so $h(0) = f + e$ or $h(0) = f - e$ for some $e \in Id(R)$ and $f \in K(R)$. Define an element $g \in T$ by,

$$g(n) = \begin{cases} f & n = 0 \\ h(n) & n > 0. \end{cases}$$

Then $h = g + h_e$ or $h = g - h_e$, where $g \in K(T)$ and $h_e \in Id(T)$. Then $T = (HR, \alpha)$ is weakly f-clean. □

Here we shall formulate two questions of interest.

Problem 2.24. *When is a matrix ring weakly f-clean?*

Problem 2.25. *Let R be a ring and $e \in Id(R)$ such that the subring eRe is weakly f-clean. Is R also weakly f-clean?*

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