Quasigroups and Related Systems **32** (2024), 141 – 154

https://doi.org/10.56415/qrs.v32.12

On ϕ-2-absorbing primary subsemimodules over commutative semirings

Issaraporn Thongsomnuk, Ronnason Chinram
Pattarawan Singavananda and Patipat Chumket

Abstract. In this paper, we introduce the concepts of ϕ-2-absorbing primary subsemimodules over commutative semirings. Let R be a commutative semiring with identity and M be an R-semimodule. Let $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ is the set of subsemimodules of M. A proper subsemimodule N of M is said to be a ϕ-2-absorbing primary subsemimodule of M if $rsx \in N \setminus \phi(N)$ implies $rx \in N$ or $sx \in N$ or $rs \in \sqrt{(N:M)}$, where $r, s \in R$ and $x \in M$. We prove some basic properties of these subsemimodules, give a characterization of ϕ-2-absorbing primary subsemimodules, and investigate ϕ-2-absorbing primary subsemimodules of quotient semimodules.

1. Introduction

In 2007, the concept of 2-absorbing ideals of rings was introducted by Badawi [3]. He defined *a 2-absorbing ideal* I of a commutative ring R to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Later in 2011 [7], Darani and Soheilnia introduced the concept of 2-absorbing submodules and studied their properties. A proper submodule N of an R-module M is said to be a *2-absorbing submodule of* M if $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$.

In 2012, Chaudhari introduced the concept of 2-absorbing ideals of a commutative semiring in [6]. He defined a *2-absorbing ideal* I of a commutative semiring R to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In the same year, Thongsomnuk

2010 Mathematics Subject Classification: 13C05, 13C13, 16Y60
Keywords: Semimodule, ϕ-2-absorbing primary subsemimodule, subtractive subsemimodule, Q-subsemimodule

introduced the concept of 2-*absorbing subsemimodules over commutative semirings* as a proper subsemimodule N of an R-semimodule M such that if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$. The concept of 2-absorbing ideals of commutative semirings and 2-absorbing subsemimodules has been widely recognized by several mathematicians, see [8] and [11].

Atani and Kohan (2010) introduced and examined the concept of primary ideals in a commutative semiring, as well as primary subsemimodules in semimodules over a commutative semiring (see [5]). They defined *a primary ideal I of a commutative semiring R* as a proper ideal, such that whenever $a, b \in R$ with $ab \in I$, then $a \in I$ or $b^k \in I$ for some $k \in \mathbb{N}$. Similarly, *a primary subsemimodule N of an R-semimodule M* is defined as a proper subsemimodule, such that whenever $a \in R$ and $m \in M$ with $am \in N$, then $m \in N$ or $a^k \in (N : M)$ for some $k \in \mathbb{N}$. In 2015, Dubey and Sarohe [9] defined the concept of 2-absorbing primary subsemimodules of a semimodule M over a commutative semiring R with $1 \neq 0$ which is a generalization of primary subsemimodules of semimodules. A proper subsemimodule N of a semimodule M is said to be a *2-absorbing primary subsemimodule of M* if $abm \in N$ implies $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$ for some $a, b \in R$ and $m \in M$.

Anderson and Batanieh (2008) generalized the concept of prime ideals, weakly prime ideals, almost prime ideals, n-almost prime ideals and ω-prime ideals of rings to ϕ-prime ideals of rings with ϕ, see in [1]. They defined a *ϕ-prime ideal* I of a ring R with ϕ be a proper ideal and if for $a, b \in R$, $ab \in I \setminus \phi(I)$ implies $a \in I$ or $b \in I$. Later in 2016, Petchkaew, Wasanawichit and Pianskool [13] introduced the concept of ϕ-n-absorbing ideals which are a generalization of n-absorbing ideals. A proper ideal I of R is called a *ϕ-n-absorbing ideal* if whenever $x_1, x_2, \ldots, x_{n+1} \in I \setminus \phi(I)$ for $x_1, x_2, \ldots x_{n+1} \in R$, then $x_1 x_2 \ldots x_{i-1} x_{i+1} \ldots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. In 2017, Moradi and Ebrahimpour [12] introduced the concept of ϕ-2-absorbing primary and ϕ-2-absorbing primary submodules. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ is the set of $R-$module M. They said that a proper submodule N of M is a *ϕ-2-absorbing primary submodule* if $rsx \in N \setminus \phi(N)$ implies $rx \in N$, or $sx \in N$, or $rs \in \sqrt{(N : M)}$, where $r, s \in R$ and $x \in M$.

In this paper, we extend the concepts of ϕ-2-absorbing primary submodules over commutative rings to the concepts of ϕ-2-absorbing primary subsemimodules over commutative semirings. We explore fundamental properties of these subsemimodules, provide a characterization of ϕ-2-absorbing primary subsemimodules, and investigate ϕ-2-absorbing primary subsemimodules of quotient semimodules.

2. Preliminaries

Definition 2.1. [10] Let R be a semiring. A *left R-semimodule* (or a left semimodule over R) is a commutative monoid $(M, +)$ with additive identity 0_M for which a function $R \times M \to M$, denoted by $(r, m) \mapsto rm$ and called the scalar multiplication, satisfies the following conditions for all elements r and r' of R and all elements m and m' of M:

(1) $(rr')m = r(r'm)$,

(2) $r(m + m') = rm + rm'$,

(3) $(r + r')m = rm + r'm$,

(4) $1_R m = m$, and

(5) $r0_M = 0_M = 0_R m$.

Throughout this paper, we assume that R is a commutative semirings identity $1 \neq 0$ and a left R-semimodule will be considered as a unitary semimodule.

Definition 2.2. [10] Let M be an R-semimodule and N a subset of M. We say N is a *subsemimodule of M* precisely when N is itself an R-semimodule with respect to the operations for M.

Definition 2.3. [5] Let M be an R-semimodule, N a subsemimodule of M, and $m \in M$. Then an *associated ideal* of N is denoted as
$(N : M) = \{r \in R \mid rM \subseteq N\}$ and $(N : m) = \{r \in R \mid rm \in N\}$.

Definition 2.4. [5] An ideal I of a semiring R is called a *subtractive ideal* if $a, a + b \in I$ and $b \in R$, then $b \in I$.

A subsemimodule N of an R-semimodule M is called a *subtractive subsemimodule* if $x, x + y \in N$ and $y \in M$, then $y \in N$.

Proposition 2.5. [5] *Let M be an R-semimodule. If N is a subtractive subsemimodule of M and $m \in M$, then $(N : M)$ and $(N : m)$ are subtractive ideals of R.*

Lemma 2.6. *Let $(N : M)$ be a subtractive ideal of R. If $a \in (N : M)$ and $a + b \in \sqrt{(N : M)}$, then $b \in \sqrt{(N : M)}$.*

Proof. Assume that $a \in (N : M)$ and $a + b \in \sqrt{(N : M)}$. There exists $k \in \mathbb{N}$ such that $(a + b)^k \in (N : M)$. Then $\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i \in (N : M)$. Since $\sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^i \in (N : M)$ and $(N : M)$ is a subtractive ideal, we obtain $b^k \in (N : M)$. Thus, $b \in \sqrt{(N : M)}$. □

Definition 2.7. [12] Let M be an R-semimodule. We define the functions $\phi_\alpha : S(M) \to S(M) \cup \{\emptyset\}$ as follows: $\phi_0(N) = 0$, $\phi_\emptyset(N) = \emptyset$, $\phi_{m+1}(N) = (N : M)^m N$ for every $m \geqslant 0$ and $\phi_\omega(N) = \bigcap_{m=0}^{\infty} (N : M)^m N$, where N is a subsemimodule of M and $S(M)$ is the set of subsemimodules of M.

Definition 2.8. [12] Let M be an R-semimodule, $S(M)$ the set of subsemimodules of M and let $f_1, f_2 \colon S(M) \to S(M) \cup \{\emptyset\}$ be two functions. Then $f_1 \leqslant f_2$ if $f_1(N) \subseteq f_2(N)$ for all $N \in S(M)$.

Definition 2.9. [2] A subsemimodule N of an R-semimodule M is called a *partitioning subsemimodule*(or *Q-subsemimodule*) if there exists a nonempty subset Q of M such that

1. $RQ \subseteq Q$ where $RQ = \{rq | r \in R \text{ and } q \in Q\}$,
2. $M = \cup\{q + N | q \in Q\}$ where $q + N = \{q + n | n \in N\}$, and
3. if $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset$ if and only if $q_1 = q_2$.

Let M be an R-semimodule and N a Q-subsemimodule of M. Let $M/N_{(Q)} = \{q + N | q \in Q\}$. Then $M/N_{(Q)}$ is a semimodule over R under the addition $\oplus$ and the scalar multiplication $\odot$ defined as follow: for any $q_1, q_2, q \in Q$ and $r \in R$, $(q_1+N) \oplus (q_2+N) = q_3+N$ and $r \bigodot (q+N) = q_4+N$ where $q_3, q_4 \in Q$ are the unique elements such that $q_1 + q_2 + N \subseteq q_3 + N$ and $rq + N \subseteq q_4 + N$. The R-semimodule $M/N_{(Q)}$ is called the *quotient semimodule of M by N*.

Lemma 2.10. [4] *Let M be an R-semimodule, N a Q-subsemimodule of M and P a subtractive subsemimodule of M with $N \subseteq P$. Then the followings hold:*

1. *N is a $Q \cap P$-subsemimodule of P.*
2. *$P/N_{(Q\cap P)} = \{q + N | q \in Q \cap P\}$ is a subsemimodule of $M/N_{(Q)}$.*

Remark 2.11. *The zero element of $P/N_{Q\cap P}$ is the same as the zero element of $M/N_{(Q)}$ which is $0_M + N$.*

3. ϕ-2-absorbing primary subsemimodules

In this section, we investigate the ϕ-2-absorbing primary subsemimodules over commutative semirings. Initially, we introduce a novel definition for ϕ-2-absorbing primary subsemimodules. Subsequently, we explore various properties of ϕ-2-absorbing primary subsemimodules.

Definition 3.1. Let M be an R-semimodule, $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function, where $S(M)$ is the set of subsemimodules of M. We say a proper subsemimodule N of M is a *ϕ-2-absorbing primary subsemimodule* if whenever $rsx \in N \backslash \phi(N)$ implies $rx \in N$, or $sx \in N$, or $rs \in \sqrt{(N : M)} = \{a \in R \mid a^n M \subseteq N \text{ for some } n \in \mathbb{N}\}$, where $r, s \in R$ and $x \in M$.

Theorem 3.2. *Let M be an R-semimodule, N a ϕ-2-absorbing primary subsemimodule of M and K be a subsemimodule of M such that $\phi(N \cap K) = \phi(N)$. Then $N \cap K$ is a ϕ-2-absorbing primary subsemimodule of K.*

Proof. Clearly, $N \cap K$ is a proper subsemimodule of K. Let $rsx \in (N \cap K) \setminus \phi(N \cap K)$ where $r, s \in R$ and $x \in K$. We have $rsx \in N \setminus \phi(N \cap K)$. Thus, $rsx \in N \setminus \phi(N)$ because $\phi(N \cap K) = \phi(N)$. Since N is a ϕ-2-absorbing primary subsemimodule of M, we obtain $rx \in N$, or $sx \in N$, or $rs \in \sqrt{(N : M)}$. If $rx \in N$ or $sx \in N$, then $rx \in N \cap K$ or $sx \in N \cap K$ because $x \in K$ and K is an R-semimodule. If $rs \in \sqrt{(N : M)}$, then $(rs)^n M \subseteq N$ for some positive integer n. In particular, $(rs)^n K \subseteq (rs)^n M \subseteq N$ and we know that $(rs)^n K \subseteq K$. Then $(rs)^n K \subseteq N \cap K$ for some positive integer n. Thus, $rs \in \sqrt{(N \cap K : K)}$. Hence $N \cap K$ is a ϕ-2-absorbing primary subsemimodule of K. $\square$

Consider the following example. Let $R = \mathbb{Z}_0^+$ and $M = \mathbb{Z}_0^+$, where throughout this paper, $\mathbb{Z}_0^+$ denotes the set of non-negative integers (including zero). We define the function $\phi : S(\mathbb{Z}_0^+) \to S(\mathbb{Z}_0^+) \cup \{\emptyset\}$ by $\phi(A) = \{0\}$ where $A \in S(\mathbb{Z}_0^+)$. Clearly, $8\mathbb{Z}_0^+$ is a ϕ-2-absorbing primary subsemimodule of $\mathbb{Z}_0^+$ and $m\mathbb{Z}_0^+$ is a subsemimodule of $\mathbb{Z}_0^+$ where $m \in \mathbb{Z}_0^+$. We see that $\phi(8\mathbb{Z}_0^+ \cap m\mathbb{Z}_0^+) = \{0\} = \phi(8\mathbb{Z}_0^+)$. Then $8\mathbb{Z}_0^+ \cap m\mathbb{Z}_0^+ = [8, m]\mathbb{Z}_0^+$ is a ϕ-2-absorbing primary subsemimodule of $m\mathbb{Z}_0^+$. This example demonstrates the concept outlined in Theorem 3.13.

Theorem 3.3. *Let M be an R-semimodule, $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\phi\}$ a function, and let N be a proper subsemimodule of M. Then the following conditions are equivalent:*

1. *N is a ϕ-2-absorbing primary subsemimodule of M.*

2. *For every $r \in R$ and $x \in M$ with $rx \notin N$,*

$$(N : rx) \subseteq (\sqrt{(N : M)} : r) \cup (N : x) \cup (\phi(N) : rx).$$

Proof. First, let $a \in (N : rx)$. Then $arx \in N$. If $arx \in \phi(N)$, then $a \in (\phi(N) : rx)$. If $arx \notin \phi(N)$, then $arx \in N \setminus \phi(N)$. Since N is a ϕ-2-absorbing primary subsemimodule of M and $rx \notin N$, we have $ax \in N$ or $a \in (\sqrt{(N : M)} : r)$. Hence $(N : rx) \subseteq (\sqrt{(N : M)} : r) \cup (N : x) \cup (\phi(N) : rx)$.

Conversely, let $r, s \in R$ and $x \in M$ with $rsx \in N \setminus \phi(N)$ and $rx \notin N$. Since $rsx \in N$ and $rsx \notin \phi(N)$, we obtain $s \in (N : rx)$ and $s \notin (\phi(N) : rx)$. From $(N : rx) \subseteq (\sqrt{(N : M)} : r) \cup (N : x) \cup (\phi(N) : rx)$. Thus, $s \in (\sqrt{(N : M)} : r)$ or $s \in (N : x)$. Hence, $sr \in \sqrt{(N : M)}$ or $sx \in N$. Therefore, N is a ϕ-2-absorbing primary subsemimodule of M. □

Moradi and Ebrahimpour [12] introduced the definition of ϕ-triple-zero within the context of submodules. In this work, we will extend and adapt this definition to apply specifically to subsemimodules.

Definition 3.4. Let M be an R-semimodule, and $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function. Assume that N is a ϕ-2-absorbing primary subsemimodule of M, $r, s \in R$ and $x \in M$. We say (r, s, x) is a *ϕ-triple-zero* of N if $rsx \in \phi(N), rx, sx \notin N$ and $rs \notin \sqrt{(N : M)}$.

Theorem 3.5. *Let M be an R-semimodule, $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function, and let N be a subtractive subsemimodule of M such that $\phi(N) \subseteq N$. Assume that N is a ϕ-2-absorbing primary subsemimodule of M and (r, s, x) is a ϕ-triple-zero of N. Then the following statements hold:*

1. *$r(N : M)x \subseteq \phi(N)$ and $s(N : M)x \subseteq \phi(N)$.*

2. *$(N : M)^2 x \subseteq \phi(N)$.*

3. *$rsN \subseteq \phi(N)$.*

4. *$r(N : M)N \subseteq \phi(N)$ and $s(N : M)N \subseteq \phi(N)$.*

Proof. (1). Suppose that there exists $t \in (N : M)$ such that $rtx \notin \phi(N)$. Since (r, s, x) is a ϕ-triple-zero of N, we have $rsx \in \phi(N)$. So, $r(s + t)x = rsx + rtx \notin \phi(N)$. Since $\phi(N) \subseteq N$, we obtain $r(s + t)x \in N \setminus \phi(N)$.

Since N is a ϕ-2-absorbing primary subsemimodule of M and $rx, sx \notin N$, we have $r(t+s) \in \sqrt{(N:M)}$. By Lemma 2.6 and $rt \in (N:M)$, we have $rs \in \sqrt{(N:M)}$, which is a contradiction with ϕ-triple-zero of N. Therefore, $r(N:M)x \subseteq \phi(N)$. Similarly, $s(N:M)x \subseteq \phi(N)$.

(2). Suppose that there exists $t, k \in (N:M)$ such that $tkx \notin \phi(N)$. Since (r,s,x) is a ϕ-triple-zero of N, we have $rsx \in \phi(N)$. By part (1), we have $stx, rkx \in \phi(N)$. Thus, $(t+r)(k+s)x \notin \phi(N)$. Then $(t+r)(k+s)x \in N \backslash \phi(N)$. Since N is a ϕ-2-absorbing primary subsemimodule of M and $rx, sx \notin N$, we have $(t+r)(k+s) \in \sqrt{(N:M)}$. By Lemma 2.6, we obtain $rs \in \sqrt{(N:M)}$, which is a contradiction with ϕ-triple-zero of N. Hence, $(N:M)^2 x \subseteq \phi(N)$.

(3). Suppose that there exists $y \in N$ such that $rsy \notin \phi(N)$. Since (r,s,x) is a ϕ-triple-zero of N, we have $rsx \in \phi(N)$. So, $rs(x+y) \notin \phi(N)$. Then $rs(x+y) \in N \backslash \phi(N)$ because $\phi(N) \subseteq N$. Since N is a ϕ-2-absorbing primary subsemimodule, $r(x+y) \in N$ or $s(x+y) \in N$ or $rs \in \sqrt{(N:M)}$. Since N is a subtractive subsemimodule and $y \in N$, we obtain $rx \in N$ or $sx \in N$ or $rs \in \sqrt{(N:M)}$, which is a contradiction with ϕ-triple-zero of N. Therefore, $rsN \subseteq \phi(N)$.

(4). Suppose that there exists $t \in (N:M)$ and $y \in N$ such that $rty \notin \phi(N)$. Since (r,s,x) is a ϕ-triple-zero of N, we obtain $rsx \in \phi(N)$. By parts (1) and (3), we have $rtx, rsy \in \phi(N)$. So, $r(s+t)(x+y) \notin \phi(N)$. Since $\phi(N) \subseteq N$ and $y \in N$, we get $r(s+t)(x+y) \in N \backslash \phi(N)$. Since N is a ϕ-2-absorbing primary subsemimodule, $r(x+y) \in N$ or $(s+t)(x+y) \in N$ or $r(s+t) \in \sqrt{(N:M)}$. Since N is a subtractive subsemimodule and Lemma 2.6, we have $rx \in N$ or $sx \in N$ or $rs \in \sqrt{(N:M)}$, which is a contradiction with ϕ-triple-zero of N. Hence, $r(N:M)N \subseteq \phi(N)$. Similarly, $s(N:M)N \subseteq \phi(N)$. □

Corollary 3.6. *Let M be an R-semimodule, $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function, and let N be a subtractive subsemimodule of M such that $\phi(N) \subseteq N$. Assume that N is a ϕ-2-absorbing primary subsemimodule of M and is not a 2-absorbing primary subsemimodule. Then $(N:M)^2 N \subseteq \phi(N)$.*

Proof. Since N is a ϕ-2-absorbing primary subsemimodule of M and is not a 2-absorbing primary subsemimodule, we have (r,s,x) is a ϕ-triple-zero of N. Assume that $t, k \in (N:M)$, $y \in N$ and $tky \notin \phi(N)$. So, $tky \in N \backslash \phi(N)$. Consider $(r+t)(s+k)(x+y) \notin \phi(N)$ because N is a ϕ-triple zero and Theorem 3.5 and $\phi(N) \subseteq N$ is subtractive subsemimodule. Then $(r+t)(s+k)(x+y) \in N \backslash \phi(N)$. Since N is a ϕ-2-absorbing primary subsemimodule, we have $(r+t)(x+y) \in N$ or $(s+k)(x+y) \in N$ or $(r+t)(s+k) \in \sqrt{(N:M)}$. Since N is a subtractive subsemimodule and Lemma 2.6, we have $rx \in N$ or $sx \in N$ or $rs \in \sqrt{(N:M)}$, which is a contradiction with ϕ-triple-zero of N. Therefore, $(N:M)^2N \subseteq \phi(N)$. $\square$

To illustrate Theorem 3.16(3), consider the following example. We define a function $\phi : S(\mathbb{Z}_0^+) \to S(\mathbb{Z}_0^+) \cup \{\emptyset\}$ by $\phi(A) = 2A$ where $A \in S(\mathbb{Z}_0^+)$. In this context, $15\mathbb{Z}_0^+$ is demonstrably a ϕ-2-absorbing primary subsemimodule and a subtractive subsemimodule of $\mathbb{Z}_0^+$. Interestingly, $30\mathbb{Z}_0^+ = \phi(15\mathbb{Z}_0^+) \subseteq 15\mathbb{Z}_0^+$. Furthermore, the triplet $(3, 10, 2)$ qualifies as a ϕ-triple-zero of $15\mathbb{Z}_0^+$. In this case, $(3 \cdot 10) \cdot 15\mathbb{Z}_0^+ = 450\mathbb{Z}_0^+ \subseteq 30\mathbb{Z}_0^+$, which aligns with the concept outlined in Theorem 3.16(3).

In 2017, the concept of weakly ϕ-2-absorbing primary submodules was introduced by Moradi and Ebrahimpour [12]. In the current study, we will extend this idea and provide a definition for weakly ϕ-2-absorbing primary subsemimodules.

Definition 3.7. Let M be an R-semimodule, $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ is the set of $R-$module M. They said that a proper submodule N of M is a *weakly ϕ-2-absorbing primary submodule* if $0 \neq rsx \in N \setminus \phi(N)$ implies $rx \in N$, or $sx \in N$, or $rs \in \sqrt{(N:M)}$, where $r, s \in R$ and $x \in M$.

Proposition 3.8. *Let M be an R-semimodule, $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function, and let N be subtractive subsemimodule of M such that $\phi(N) \subseteq N$ that is not 2-absorbing primary subsemimodule of M. If N is a weakly 2-absorbing primary subsemimodule of M, then $(N:M)^2N = \{0\}$.*

Proof. Assume that N is a weakly 2-absorbing primary subsemimodule of M but N is not 2-absorbing primary subsemimodule of M. Then N is a ϕ_0-2-absorbing primary subsemimodule of M. By Corollary 3.6, we obtain $(N:M)^2N \subseteq \phi_0(N) = \{0\}$. Clearly, $\{0\} \subseteq (N:M)^2N$. Thus, $(N:M)^2N = \{0\}$. $\square$

Subsequently, we analyze the function ϕ_n, as defined in Definition 2.7, for cases where $n \leqslant 4$. We also explore the function ϕ_ω, also defined in Definition 2.7, which establishes a connection with ϕ-2-absorbing primary subsemimodules.

Proposition 3.9. *Let M be an R-semimodule, $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function, and let N be subtractive subsemimodule of M such that*

$\phi(N) \subseteq N$ that is not 2-absorbing primary subsemimodule of M. If N is a ϕ-2-absorbing primary subsemimodule of M for some ϕ with $\phi \leqslant \phi_4$, then $(N : M)^2N = (N : M)^3N$.

Proof. Assume that N is a ϕ-2-absorbing primary subsemimodule of M with $\phi \leqslant \phi_4$ and N is not 2-absorbing primary subsemimodule. By Corollary 3.6, we obtain $(N : M)^2N \subseteq \phi(N)$. Since $\phi \leqslant \phi_4$, then $\phi(N) \subseteq \phi_4(N) = (N : M)^3N$. Now, we have $(N : M)^2N \subseteq (N : M)^3N$. Since N is an R-semimodule, we have $(N : M)^3N = (N : M)(N : M)^2N \subseteq (N : M)^2N$. Therefore, $(N : M)^2N = (N : M)^3N$. □

Corollary 3.10. *Let M be an R-semimodule, $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function, and let N be subtractive subsemimodule of M such that $\phi(N) \subseteq N$. If N is a ϕ-2-absorbing primary subsemimodule of M with $\phi \leqslant \phi_4$, then N is a ϕ_ω-2-absorbing primary subsemimodule of M.*

Proof. Assume that N is a ϕ-2-absorbing primary subsemimodule of M with $\phi \leqslant \phi_4$. It's clear that N is a ϕ_ω-2-absorbing primary subsemimodule of M if N is a 2-absorbing primary subsemimodule. Now, we consider in case that N is not 2-absorbing primary, then $(N : M)^2N = (N : M)^3N$, by Proposition 3.9. Since N is a ϕ-2-absorbing primary subsemimodule of M with $\phi \leqslant \phi_4$, we have N is ϕ_4-2-absorbing primary. So, $\phi_\omega(N) = \bigcap_{m=0}^{\infty}(N : M)^mN = (N : M)^3N = \phi_4$. Thus, N is a ϕ_ω-2-absorbing primary subsemimodule of M. □

Lemma 3.11. *Let N be a subtractive ϕ-2-absorbing primary subsemimodule of an R-semimodule M and $a, b \in R$. Suppose that $abK \subseteq N\backslash\phi(N)$ for some subsemimodule K of M. Then $ab \in \sqrt{(N : M)}$ or $aK \subseteq N$ or $bK \subseteq N$.*

Proof. Let $abK \subseteq N \setminus \phi(N)$ for some subsemimodule K of M. Assume that $ab \notin \sqrt{(N : M)}$, $aK \nsubseteq N$ and $bK \nsubseteq N$. Then $ak_1 \notin N$ and $bk_2 \notin N$ for some $k_1, k_2 \in K$. Since $abk_1 \in N\backslash\phi(N)$, $ab \notin \sqrt{(N : M)}$, $ak_1 \notin N$ and N is a ϕ-2-absorbing primary subsemimodule, we have $bk_1 \in N$. Since $abk_2 \in N\backslash\phi(N), ab \notin \sqrt{(N : M)}, bk_2 \notin N$ and N is a ϕ-2-absorbing primary subsemimodule, we obtain $ak_2 \in N$. We know that $ab(k_1 + k_2) \in N\backslash\phi(N)$ and $ab \notin \sqrt{(N : M)}$. Since N is a ϕ-2-absorbing primary subsemimodule, we have $a(k_1 + k_2) \in N$ or $b(k_1 + k_2) \in N$. If $a(k_1 + k_2) \in N$, then $ak_1 \in N$ (as N is a subtractive), which is a contradiction. If $b(k_1+k_2) \in N$, then $bk_2 \in N$ (as N is a subtractive), which is a contradiction. Hence, $ab \in \sqrt{(N : M)}$ or $aK \subseteq N$ or $bK \subseteq N$. □

Theorem 3.12. *Let K be a subtractive subsemimodule of M and $\sqrt{(K:M)}$ be a subtractive ideal of R. If K is a ϕ-2-absorbing primary subsemimodule of M, then whenever $IJN \subseteq K\backslash\phi(K)$ for some ideals I, J of R and a subsemimodule N of M, then $IJ \subseteq \sqrt{(K:M)}$ or $IN \subseteq K$ or $JN \subseteq K$.*

Proof. Let K be a ϕ-2-absorbing primary subsemimodule of M. Assume that $IJN \subseteq K\backslash\phi(K)$ for some ideals I, J of R and a subsemimodule N of M. Suppose that $IJ \nsubseteq \sqrt{(K:M)}$, $IN \nsubseteq K$ and $JN \nsubseteq K$. Then $a_1N \nsubseteq K$ and $b_1N \nsubseteq K$ for some $a_1 \in I$ and $b_1 \in J$. Since $a_1b_1N \subseteq K\backslash\phi(K), a_1N \nsubseteq K, b_1N \nsubseteq K$ and Lemma 3.11, we have $a_1b_1 \in \sqrt{(K:M)}$. Since $IJ \nsubseteq \sqrt{(K:M)}$, we have $a_2b_2 \notin \sqrt{(K:M)}$ for some $a_2 \in I$ and $b_2 \in J$. Since $a_2b_2N \subseteq K\backslash\phi(K)$ and $a_2b_2 \notin \sqrt{(K:M)}$, we have $a_2N \subseteq K$ or $b_2N \subseteq K$ by Lemma 3.11. Here three cases arise.

Case I: When $a_2N \subseteq K$ but $b_2N \nsubseteq K$. Since $a_1b_2N \subseteq K\backslash\phi(K)$, $b_2N \nsubseteq K$ and $a_1N \nsubseteq K$, then by Lemma 3.11, $a_1b_2 \in \sqrt{(K:M)}$. We know that $a_2N \subseteq K$ but $a_1N \nsubseteq K$, so $(a_1+a_2)N \nsubseteq K$ (as K is subtractive). Since $(a_1+a_2)b_2N \subseteq K\backslash\phi(K)$, $b_2N \nsubseteq K$ and $(a_1+a_2)N \nsubseteq K$, we have $(a_1+a_2)b_2 \in \sqrt{(K:M)}$ by Lemma 3.11. Since $a_1b_2 \in \sqrt{(K:M)}$ and $\sqrt{(K:M)}$ is subtractive, we have $a_2b_2 \in \sqrt{(K:M)}$, which is a contradiction.

Case II: When $b_2N \subseteq K$ but $a_2N \nsubseteq K$. We can conclude similary to Case I.

Case III: When $a_2N \subseteq K$ and $b_2N \subseteq K$. Since $b_2N \subseteq K$ and $b_1N \nsubseteq K$, we have $(b_1+b_2)N \nsubseteq K$. Since $a_1(b_1+b_2)N \subseteq K\backslash\phi(K)$, $(b_1+b_2)N \nsubseteq K$ and $a_1N \nsubseteq K$, we get that $a_1(b_1+b_2) \in \sqrt{(K:M)}$ by Lemma 3.11. Since $a_1b_1 \in \sqrt{(K:M)}$ and $\sqrt{(K:M)}$ is subtractive, we conclude that $a_1b_2 \in \sqrt{(K:M)}$. Since $a_2N \subseteq K$, $a_1N \nsubseteq K$ and K is subtractive implies $(a_1+a_2)N \nsubseteq K$. Since $(a_1+a_2)b_1N \subseteq K\backslash\phi(K)$, $(a_1+a_2)N \nsubseteq K$ and $b_1N \nsubseteq K$, we have $(a_1+a_2)b_1 \in \sqrt{(K:M)}$ by Lemma 3.11. Since $a_1b_1 \in \sqrt{(K:M)}$, $(a_1+a_2)b_1 \in \sqrt{(K:M)}$ and $\sqrt{(K:M)}$ is subtractive, we have $a_2b_1 \in \sqrt{(K:M)}$. Since $(a_1+a_2)(b_1+b_2)N \subseteq K\backslash\phi(K)$, $(a_1+a_2)N \nsubseteq K$ and $(b_1+b_2)N \nsubseteq K$, by Lemma 3.11, $(a_1+a_2)(b_1+b_2) \in \sqrt{(K:M)}$. Since $a_2b_1, a_1b_2, a_1b_1 \in \sqrt{(K:M)}$ and $\sqrt{(K:M)}$ is subtractive, then $a_2b_2 \in \sqrt{(K:M)}$, which is a contradiction.

Hence, $IJ \subseteq \sqrt{(K:M)}$ or $IN \subseteq K$ or $JN \subseteq K$. □

Theorem 3.13. *Let M an R-semimodule, and let $\phi : S(M) \longrightarrow S(M)\cup\{\emptyset\}$ be a function. Assume that N is a subsemimodule of M such that $\phi(N)$ is a*

2-absorbing primary subsemimodule of M and $\phi(N) \subseteq N$. Then N is a ϕ-2-absorbing primary subsemimodule of M if and only if N is a 2-absorbing primary subsemimodule of M.

Proof. First, assume that N is a ϕ-2-absorbing primary subsemimodule of M and $\phi(N)$ is a 2-absorbing primary subsemimodule of M. Let $r, s \in R$ and $x \in M$ with $rsx \in N$. Suppose that neither rx nor sx is in N. Here two cases arise.

Case I: $rsx \in \phi(N)$. Then $rs \in \sqrt{(\phi(N) : M)} \subseteq \sqrt{(N : M)}$ because $\phi(N)$ is a ϕ-2-absorbing primary subsemimodule, $\phi(N) \subseteq N$ and $rx, sx \notin N$.

Case II: $rsx \notin \phi(N)$. Since N is a ϕ-2-absorbing primary subsemimodule and $rx, sx \notin N$, we obtain $rs \in \sqrt{(N : M)}$.

Conversely, it's clearly. □

Let M be an R-semimodule, N be a Q-subsemimodule of M. For a function $\boldsymbol{\phi} : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ we define the function $\boldsymbol{\phi_N} : S(M/N_{(Q)}) \longrightarrow S(M/N_{(Q)}) \cup \{\emptyset\}$ by $\phi_N(K/N) = \phi(K)/N_{(\phi(K)\cap Q)}$ if $\phi(K) \neq \emptyset$, and $\phi_N(K/N) = \emptyset$ if $\phi(K) = \emptyset$, for every subsemimodule K of M with $N \subseteq K$.

Theorem 3.14. *Let M be an R-semimodule, N a Q-subsemimodule of M and $P, \phi(P)$ are subtractive subsemimodules of M with $N \subseteq P$. Then P is a ϕ-2-absorbing primary subsemimodule of M if and only if $P/N_{(Q\cap P)}$ is a ϕ_N-2-absorbing primary subsemimodule of $M/N_{(Q)}$.*

Proof. First, assume that P is a ϕ-2-absorbing primary subsemimodule of M. Then we have $P/N_{(Q\cap P)}$ is a subsemimodule of $M/N_{(Q)}$. Now let $r, s \in R$ and $q_1 + N \in M/N_{(Q)}$ where $q_1 \in Q$ be such that $rs \odot (q_1 + N) \in P/N_{(Q\cap P)} \setminus \phi_N(P/N_{(Q\cap P)})$. Then there existe unique $q_2 \in Q \cap P$ such that $rs \odot (q_1 + N) = q_2 + N$ where $rsq_1 + N \subseteq q_2 + N$. Since $q_2 \in P$ and $N \subseteq P$, we have $rsq_1 + N \subseteq P$. Since $N \subseteq P$ and P is a subtractive subsemimodule, $rsq_1 \in P$. Since $rsq_1 + N \subseteq q_2 + N \notin \phi_N(P/N_{(Q\cap P)})$, we obtain $rsq_1 + N \subseteq q_2 + N \notin \phi(P)/N_{(Q\cap\phi(P))}$. Thus, we have $rsq_1 = q_2 + x$ for some $x \in N \subseteq \phi(P)$. Since $q_2 \notin Q \cap \phi(P)$, we get $q_2 \notin \phi(P)$. Then $rsq_1 = q_2 + x \notin \phi(P)$ because $\phi(P)$ is subtractive. Now, we have $rsq_1 \in P \setminus \phi(P)$. Since P is a ϕ-2-absorbing subsemimodule of M, it can be concluded that $rq_1 \in P$ or $sq_1 \in P$ or $rs \in \sqrt{(P : M)}$. We claim that $r \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $s \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $rs \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)}}$.

Case I: $rq_1 \in P$. Since $q_1 \in Q$, we have $rq_1 \in Q$. Then $rq_1 \in Q \cap P$. So, $rq_1 + N \in P/N_{(Q\cap P)}$. Moreover, $r \odot (q_1 + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $rq_1 + N \subseteq q_3 + N$. Then $rq_1 = q_3 + x_1$ for some $x_1 \in N \subseteq P$. Since P is subtractive, we have $q_3 \in P$. Thus, $r \odot (q_1 + N) = q_3 + N \in P/N_{(Q\cap P)}$.

Case II: $sq_1 \in P$. We can conclude similarly to Case I that $s \odot (q_1 + N) \in P/N_{(Q\cap P)}$.

Case III: $rs \in \sqrt{(P : M)}$. Then there exists $k \in \mathbb{N}$ such that $(rs)^k \in (P : M)$. So, $(rs)^k M \subseteq P$. Let $q + N \in M/N_{(Q)}$ where $q \in Q$. Consider $(rs)^k \odot (q + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $(rs)^k + N \subseteq q_4 + N$. So, $(rs)^k q = q_4 + x_2$ for some $x_2 \in N \subseteq P$. Since $(rs)^k \in (P : M)$, we have $(rs)^k q \in P$. Hence, $q_4 \in P$ because P is subtractive. Then $q_4 \in Q \cap P$. Thus, $(rs)^k \odot (q + N) = q_4 + N \in P/N_{(Q\cap P)}$. Hence, $rs \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)}}$.

Therefore, $P/N_{(Q\cap P)}$ is a ϕ_N-2-absorbing primary subsemimodule of $M/N_{(Q)}$.

Conversely, assume that $P/N_{(Q\cap P)}$ is a ϕ_N-2-absorbing primary subsemimodule of M. Let $r, s \in R$ and $x \in M$ such that $rsx \in P \setminus \phi(P)$. Since N is a Q-subsemimodule of M and $x \in M$, we have $x \in q_1 + N$ where $q_1 \in Q$. So, $rsx \in rs \odot (q_1 + N)$. Let $rs \odot (q_1 + N) = q_2 + N$ where q_2 is the unique element of Q such that $rsq_1 + N \subseteq q_2 + N$. Then $rsx \in q_2 + N$. So there is $y \in N$ such that $q_2 + y = rsx \in P$. Since $y \in N \subseteq P$ and P is subtractive, we obtain $q_2 \in P$. Then $q_2 \in Q \cap P$. Thus, $rs \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)}$. Consider $rsx \notin \phi(P)$ and $y \in N \subseteq \phi(P)$. Since $rsx = q_2 + y$ and $\phi(P)$ is subsemimodule, we have $q_2 \notin \phi(P)$ so that $q_2 + N \notin \phi(P)/N_{(Q\cap\phi(P))} = \phi_N(P/N)$. Now, we have $rs \odot (q_1 + N) = q_2 + N \notin P/N_{(Q\cap P)} \setminus \phi_N(P/N)$. Since $P/N_{(Q\cap P)}$ is a ϕ_N-2-absorbing primary subsemimodule of $M/N_{(Q)}$, we get $r \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $s \odot (q_1 + N) \in P/N_{(Q\cap P)}$ or $rs \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)})}$. Here three cases arise.

Case I: $r \odot (q_1 + N) \in P/N_{(Q\cap P)}$. Then $r \odot (q_1 + N) = q_2 + N$ where q_2 is the unique element of $Q \cap P$ such that $rq_1 + N \subseteq q_2 + N$. Thus, $rq_1 + N \subseteq q_2 + N \subseteq P$ because $N \subseteq P$ and $q_2 \in Q \cap P$. So, $x \in q_1 + N$ that $rx \in r(q_1 + N) \subseteq rq_1 + N \subseteq q_2 + N \subseteq P$. Thus, $rx \in P$.

Case II: $s \odot (q_1 + N) \in P/N_{(Q\cap P)}$. We can conclude similarly to Case I that $sx \in P$.

Case III: $rs \in \sqrt{(P/N_{(Q\cap P)} : M/N_{(Q)})}$. Then $(rs)^k \odot M/N_{(Q)} \subseteq P/N_{(Q\cap P)}$ for some $k \in \mathbb{N}$. Let $m \in M$. So, there is unique $q_3 \in Q$ such that $m \in q_3 + N$ and $(rs)^k m \in (rs)^k(q_3 + N) \subseteq (rs)^k \odot (q_3 + N) = q_4 + N$ where q_4 is the unique element of Q such that $(rs)^k q_3 + N \subseteq q_4 + N$. Now, $q_4 + N = (rs)^k \odot (q_3 + N) \in P/N_{(Q\cap P)}$. Then $(rs)^k m \in q_4 + N \subseteq P$. So, $(rs)^k M \subseteq P$. Thus, $(rs)^k M \subseteq P$. Therefore, $rs \in \sqrt{(P : M)}$.

Hence, P is a ϕ-2-absorbing primary subsemimodule of M. □

Corollary 3.15. *Let M be an R-semimodule, N a Q-subsemimodule of M, and let P and $\phi(P)$ be subtractive subsemimodules of M with $N \subseteq P$. If $\phi(P) = N$ and P is a ϕ-2-absorbing primary subsemimodule of M, then $P/N_{(Q\cap P)}$ is a weakly 2-absorbing primary subsemimodule of $M/N_{(Q)}$.*

Proof. Since $\phi(P) = N$, we have $\phi_N(P/N) = \phi(P)/N = \{0\}$. By Theorem 3.14, we conclude that $P/N_{(Q\cap P)}$ is a weakly 2-absorbing primary subsemimodule of $M/N_{(Q)}$. □

References

[1] **D.D. Anderson and M. Batanieh**, *Generalizations of prime ideals*, Comm. Algebra, **36** (2008), 686 – 696.

[2] **R.E. Atani, and S.E. Atani**, *On subsemimodules of semimodules*, Bul. Acad. Siinte Repub. Mold. Mat., **63** (2010), no. 2, 20 – 30.

[3] **A. Badawi**, *On 2-absorbing ideals of commutative rings*, Bull. Aust. Math. Soc., **75** (2007), no. 3, 417 – 429.

[4] **J.N. Chaudhari and B.R. Bonde**, *On partitioning and subtractive subsemimodules of semimodules over semirings*, Kyungpook Math. J., **50** (2010), 329 – 336.

[5] **J.N. Chaudhari and B.R. Bonde**, *Weakly prime subsemimodules of semimodules over semirings*, Int. J. Algebra, **5** (2011), no. 4, 167 – 174.

[6] **J.N. Chaudhari**, *2-absorbing ideals in semirings*, Int. J. Algebra, **6** (2012), no. 6, 265 – 270.

[7] **A.Y. Darani, and F. Soheilnia**, *2-absorbing and weakly 2-absorbing submodules*, Thai. J. Math., **9** (2011), no. 3, 577 – 584.

[8] **M.K. Dubey and P. Sarohe**, *On 2-absorbing semimodules*, Quasigroups Related Syst., **21** (2013), 175 – 184.

[9] **M. K. Dubey and P. Sarohe**, *On 2-absorbing primary subsemimodules over commutative semirings*, Bul. Acad. Stiinte Repub. Mold., Mat., **78** (2015), no. 2, 27 – 35.

[10] **J.S. Golan**, *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, (1999).

[11] **P. Kumar, M.K. Dubey and P. Sarohe**, *On 2-absorbing ideals in commutative semiring*, Quasigroups Related Syst. **24** (2016), 67 – 74.

[12] **R. Moradi and M. Ebrahimpour**, *On ϕ-2-absorbing primary submodule*, Acta Math. Vietnam, **42** (2017), 27 – 35.

[13] **P. Petchkaew, A. Wasanawichit, and S. Pianskool**, *Generalizations of n-absorbing ideals of commutative semirings*, Thai. J. Math., **14** (2016), no. 2, 477 – 489.

Received August 13, 2023

I. Thongsomnuk
Division of Mathematics, Faculty of Science and Technology, Phetchaburi Rajabhat University, Na Wung, Muang, Phetchaburi 76000, Thailand
E-mail: issaraporn.tho@mail.pbru.ac.th

R. Chinram
Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, 90110 Thailand
E-mail: ronnason.c@psu.ac.th

P. Singavanada
Program in Mathematics, Faculty of Science and Technology, Songkhla Rajabhat University, Khoa-Roob-Chang, Muang, Songkhla 90000, Thailand
pattarawan.pe@skru.ac.th

P. Chumket
Division of Mathematics, Faculty of Science Technology and Agriculture, Yala Rajabhat University, Tambol Sateng, Mueang, Yala 95000, Thailand
E-mail: patipat.c@yru.ac.th