Quasigroups and Related Systems **32** (2024), 255 – 259

https://doi.org/10.56415/qrs.v32.18

The structure of the orthomorphism graph of quaternion group Q_8

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Abstract. We give a very simple theoretical proof of the fact that the orthomorphism graph of quaternion group Q_8 lacks adjacency.

1. Introduction

For a group G, a bijection $\theta\colon\ G \to G$ for which the map $\phi\colon\ x \mapsto x^{-1}\theta(x)$ is also a bijection of G is called *orthomorphism* of G. Two orthomorphisms θ_1 and θ_2 of G are called *orthogonal*, written $\theta_1 \perp \theta_2$, if the map $x \mapsto \theta_1(x)^{-1}\theta_2(x)$ is a bijection from G to itself. Equivalently, $\theta_1 \perp \theta_2$ if and only if $\theta_1\theta_2^{-1}$ is an orthomorphism of G [4, Corollary 1.31, p. 22]. An orthomorphism of a group which fixes identity element of the group is called *normalized* orthomorphism. We denote the set of normalized orthomorphism of a group G by $\mathrm{Orth}(G)$. A graph in which vertices are normalized orthomorphisms of G and adjacency being synonymous with orthogonality is called an *orthomorphism graph* of G, which is also denoted by $\mathrm{Orth}(G)$. The order of the largest complete subgraph of a graph is called a *clique number* of the graph. A clique number of orthomorphism graph of a group G is denoted by $\omega(G)$.

We prove the following result:

Theorem. $\omega(Q_8) = 1$.

This problem was proposed by Evans [4, Problem 16.49]. In 1964, Chang and Tai [2] found that Q_8 has 48 normalized orthomorphisms and also reported that $\omega(Q_8) = 1$ by hand computation. This was also confirmed by Jungnickel and Grams in [6] by computer search. Later Evans and Perkel also confirmed that using Cayley (a forerunner of the computer algebra

2010 Mathematics Subject Classification: 05B15
Keywords: Orthomorphism graph

system Magma) [3]. Bedford and Whitaker [1] in 1999, first proved theoretically that Q_8 has 48 normalized orthomorphisms. In 2021, Evans [5] proved theoretically that $|\mathrm{Orth}(Q_8)| = 48$ and $\omega(Q_8) = 1$.

We give another simple proof of this fact by observing that all normalized orthomorphisms of Q_8 are conjugates of two orthomorphisms of Q_8, which are odd permutations.

An *automorphisms* A of $\mathrm{Orth}(G)$ is a bijection on $\mathrm{Orth}(G)$ such that $A(\theta_1) \perp A(\theta_2)$ if and only if $\theta_1 \perp \theta_2$ where $\theta_1, \theta_2 \in \mathrm{Orth}(G)$. For $f \in \mathrm{Aut}(G)$, the group of automorphism of G, the map $H_f\colon \mathrm{Orth}(G) \to \mathrm{Orth}(G)$ defined as $H_f(\theta) = f\theta f^{-1}$ is known as *homology* of $\mathrm{Orth}(G)$. Homology is an example of automorphism of $\mathrm{Orth}(G)$ [4, Theorem 8.6, p 206]. Any unexplained notation used in the paper is from [4].

2. Basic results

Suppose $G = \{g_1, g_2, \dots, g_n\}$ is a finite group, g_1 is the identity element of G. For $h \in \mathrm{Orth}(G)$, the map $\phi_h\colon G \to G$ defined as $\phi_h(x) = x^{-1}h(x)$ is called the *complete mapping* associated with h.

Lemma 2.1. *For $h \in \mathrm{Orth}(G)$ and $x \in G$, the following hold:*

(i) ϕ_h *is a bijection from G to itself,*

(ii) $h(x) = \phi_h(x) \iff x = 1,$

(iii) $h(x) = x \iff \phi_h(x) = 1 \iff x = 1,$

(iv) $h(x) = x^2 \iff \phi_h(x) = x.$

Proof. Follows from the definition of orthomorphism. □

Lemma 2.2. *Consider $Q_8 = \langle p, q \mid p^4 = 1, q^2 = p^2, pq = qp^{-1}\rangle$. If a bijective map $\theta\colon Q_8 \to Q_8$ is an orthomorphism, exactly* 2 *elements of the form p^i will map to elements of the form $p^j \in Q_8$.*

Proof. Follows from Lemma 2.1 in [5]. □

3. The structure of Orth(Q_8)

In this section, we will classify elements of $\mathrm{Orth}(Q_8)$ on the basis of the fixed points of associated complete mapping.

Consider $Q_8 = \{p^0, p^1, p^2, p^3, q, qp^1, qp^2, qp^3\}$.

Since Q_8 is non-cyclic 2-group, $\mathrm{Orth}(Q_8) \neq \emptyset$. Then any element of $\mathrm{Orth}(Q_8)$ will lie in exactly one of the class given below:

(i) $I_1 := \{h \in \mathrm{Orth}(Q_8) \mid h(p) = p^2\}$,

(ii) $I_2 := \{h \in \mathrm{Orth}(Q_8) \mid h(p^3) = p^2\}$,

(iii) $I_3 := \{h \in \mathrm{Orth}(Q_8) \mid h(q) = p^2\}$,

(iv) $I_4 := \{h \in \mathrm{Orth}(Q_8) \mid h(qp) = p^2\}$,

(v) $I_5 := \{h \in \mathrm{Orth}(Q_8) \mid h(qp^2) = p^2\}$,

(vi) $I_6 := \{h \in \mathrm{Orth}(Q_8) \mid h(qp^3) = p^2\}$.

Proposition 3.1.

(i) For each $i \in \{2, 3, 4, 5, 6\}$, *there exists* $x \in Aut(Q_8)$ *such that* $xI_1x^{-1} = I_i$.

(ii) $|\mathrm{Orth}(Q_8)| = 6|I_1|$.

Proof. Consider,

$x_2 = (p, p^3)(qp, qp^3)$,
$x_3 = (p, q)(qp, qp^3)(p^3, qp^2)$,
$x_4 = (p, qp)\ (q, qp^2)(p^3, qp^3)$,
$x_5 = (p, qp^2)(qp, qp^3)(q, p^3)$,
$x_6 = (p, qp^3)\ (q, qp^2)(p^3, qp)$.

Clearly $x_i \in \mathrm{Aut}(Q_8)$ and $I_i = x_iI_1x_i^{-1}$, for $2 \leqslant i \leqslant 6$. Therefore, $|I_i| = |I_1|$ for all $2 \leqslant i \leqslant 6$. Hence, $|\mathrm{Orth}(Q_8)| = 6|I_1|$. □

Proposition 3.2. *Each element of* I_1 *is an odd permutation and* $|I_1| = 8$.

Proof. Since $\mathrm{Orth}(Q_8)$ consists of normalized orthomorphisms, p^0 is fixed by elements of $\mathrm{Orth}(Q_8)$. Now, $\mathrm{Aut}(Q_8)$ is isomorphic to symmetric group on four symbols and $\mathrm{Aut}(Q_8)$ acts on $\mathrm{Orth}(Q_8)$ through conjugation. Also any automorphism of Q_8 fixes p^2, so I_1 is a block whose orbit size is 6. If B is stabilizer of I_1 under action, then

$$B = \{(), (q, qp^2)(qp, qp^3), (q, qp^3, qp^2, qp), (q, qp, qp^2, qp^3)\}.$$

For $h \in I_1$, $h(p^0) = p^0$ and $h(p) = p^2$. By Lemma 2.2, the possibility for $h(p^2)$ is q, qp, qp^2, qp^3. Clearly, there are four orbits of h under the action of B determined by the four images of $h(p^2)$. To determine all elements of I_1 means determining all elements of one orbit of the action. Suppose $h(p^2) = q$. Now, we show $h(p^3) = qp^2$. Due to bijectivity of h and ϕ_h, $h(p^3) \neq q$ and qp^3 respectively.

Assume $h(p^3) = qp$. Then by Lemma 2.2, we know that preimage of p and p^3 will be of form qp^s. Table shows possible values of ϕ_h when qp^s maps to p and p^3.

x	$\phi_h(x); h(x) = p$	$\phi_h(x); h(x) = p^3$
q	qp^3	qp
qp	q	qp^2
qp^2	qp	qp^3
qp^3	qp^2	q

From this table, it is clear that qp and qp^3 cannot map to any of p or p^3, as that will destroy the bijectivity of ϕ_h. Also, if we map q to p, then we cannot map qp^2 to p^3 and vice versa. So, our assumption is wrong, therefore, $h(p^3) = qp^2 = \phi_h(p^2)$. Again, from the table, it is clear that we have two possibilities for preimage of p, and they are q and qp. Also, fixing preimage of p will fix preimage of p^3, which finally fixes orthomorphism. So we get two orthomorphism when $p^2 \mapsto q$ and they are $h_1 = (p, p^2, q)(p^3, qp^2, qp, qp^3)$ and $h_2 = (p, p^2, q, qp^3, qp)(p^3, qp^2)$.

Clearly, one orbit has only two elements so $|I_1| = 4 \times 2 = 8$. Also, each element of I_1 is conjugate of h_1 or h_2. As h_1 and h_2 are odd permutations. Therefore, each element of I_1 is an odd permutation. □

Corollary 3.3. $|\text{Orth}(Q_8)| = 48$.

Proof. Follows from Proposition 3.1 and Proposition 3.2. □

Proof of the Theorem. We know that, if θ_1, $\theta_2 \in \text{Orth}(Q_8)$, then $\theta_1 \perp \theta_2$ if and only if $\theta_1\theta_2^{-1} \in \text{Orth}(Q_8)$. By Proposition 3.1, all elements of $\text{Orth}(Q_8)$ are conjugates of elements of I_1, and by Proposition 3.2 elements of I_1 are odd permutations, so all elements of $\text{Orth}(Q_8)$ are odd permutations. If θ_1 and θ_2 are odd permutations, then $\theta_1\theta_2^{-1}$ will be an even permutation. Since there is no even permutation in $\text{Orth}(Q_8)$, no two elements of $\text{Orth}(Q_8)$ are orthogonal. Hence, $\omega(Q_8) = 1$. □

4. Conclusion

The same method can be applied to groups G where all normalized orthomorphisms are odd permutation. It may be an interesting problem to classify those groups whose all normalized orthomorphisms are odd permutations.

Acknowledgement. We are greatful to refree(s) for their insightful comments on the paper.

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Received May 10, 2024

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