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Gyrotransversals of order p^2

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Abstract. The isomorphism classes of gyrotransversals of order p^2 is calculated in the group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$ corresponding to a fixed subgroup of order p, where p is an odd prime. As a consequence, a lower bound for the non-isomorphic right gyrogroups of order p^2 is obtained. Also, we obtain a lower bound for the non-isomorphic right gyrogroups of order p^2 of nilpotency class 2.

1. Introduction

Let H be a fixed subgroup of a group G and S be a right transversal to H in G with $e \in S$, where e is the identity of the group G. Then the set S together with the induced binary operation $\circ$ defined by $\{x \circ y\} = S \cap Hxy$ becomes a right loop with identity e (see [10]). Also, S is a right transversal in the group $\langle S \rangle$ to the subgroup $\langle S \rangle \cap H$ of $\langle S \rangle$ (see [4]). Identifying S with the set of all right cosets of H in G, we get a group homomorphism $\lambda : G \longrightarrow Sym(S)$ defined by $\{\lambda(g)(x)\} = S \cap Hxg$ for all $g \in G$ and $x \in S$. The kernel of λ is $Core_G(H)$, the core of H in G. The group $G_S = \lambda(\langle S \rangle \cap H)$ is called the group torsion of S (see [4, Definition 3.1]). Identifying S with $\lambda(S)$, we get $\lambda(\langle S \rangle) = G_S S$. Note that, the group $G_S S$ depends only on S and not on H. Also, S is a right transversal to the subgroup G_S in the group $G_S S$ (see [4]). Moreover, S is a group if and only if G_S is trivial.

Gyrogroups are special loops which are the generalization of the groups. Most of the properties of groups are shared with gyrogroups. The first example of gyrogroup structure is given by Ungar [13] which is the relativistic gyrogroup $(\mathbb{R}^3_1, \bigoplus)$ consisting of the unit ball $\mathbb{R}^3_1$ in the Euclidean 3-space $\mathbb{R}^3$ with Einstein's addition. In [1], [13] and [14] Ungar and Foguel described the properties of gyrogroups and introduced the gyrotransversals in a group to a subgroup of it (see [1, Definition 2.9]).

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In [8], Lal and Yadav studied the right gyrogroups, gyrotransversals and their deformations. They proved that right gyrogroups and gyrotransversals are same in some sense (see [1, Theorem 2.12]). In this paper, we have used the *Cauchy-Frobenius Formula* to find the number of gyrotransversals upto isomorphism in the group $G = \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$ to a fixed subgroup H of order p, where p is an odd prime. For this, we take the natural action of $Aut_H(G)$ on the set of all the gyrotransversals to the subgroup H in G, where $Aut_H(G) = \{\theta \in Aut(G) \mid \theta(H) = H\}$.

Let S and T be two right transversals to the subgroup H in G such that $\langle S\rangle = G = \langle T\rangle$ and $S \simeq T$. Then by the Proposition [7, Proposition 2.7], there exists $\theta \in Aut_H(G)$ such that $\theta(S) = T$. Also, $Aut_H(G)$ acts transitively on the set of all right transversals isomorphic to the right transversal S. Since $|H| = p$, if S is a right transversal to H in G, then either S is a subgroup of G or $\langle S\rangle = G$. Therefore, the number of orbits under the action of $Aut_H(G)$ is equal to the number of isomorphism classes of right loops. This gives us a lower bound for the number of gyrotransversals of order p^2 upto isomorphism in the group G. Lal and Yadav [8] have shown that S is a right gyrogroup precisely when S is a gyrotransversal to G_S in the group G_SS. Thus, we get a lower bound for non-isomorphic right gyrogroups of order p^2. In [11], it is proved that any gyrogroup of order p^2 is a group. But a right gyrogroup of order p^2 may not be a group. In this paper, we have found the lower bound of non-isomorphic right gyrogroups of order p^2. Also, a lower bound for the non-isomorphic right gyrogroups of nilpotency class 2 is obtained.

Throughout the paper, $\mathbb{Z}_n$ denotes the cyclic group of order n and $U(n)$ denotes the group of units of $\pmod n$. $d(n)$ denotes the set of non-trivial divisors of n, that is, not including 1 and $o(a)$ denotes the order of an element a in a group. $\phi(n)$ denotes the Euler phi function for any positive integer n.

2. Preliminaries

In this section, we give the preliminaries that we will use throughout the paper.

Definition 2.1. [1, Definition 2.3] Let $(S, \circ)$ be a groupoid with a right identity e and a right inverse a' for each element $a \in S$ such that $a \circ a' = e$. Then $(S, \circ)$ is called a right gyrogroup if,

(i) for any $x, y, z \in S$, there exists a unique element $f(y,z)(x) \in S$ such that
$$(x \circ y) \circ z = f(y,z)(x) \circ (y \circ z),$$

(ii) the map $f(y,z) : S \longrightarrow S$ given by $x \mapsto f(y,z)(x)$ is an automorphism of $(S, \circ)$,

(iii) for all $y \in S$,
$$f(y, y') = I_S.$$

By [8, Corollary 5.7], $(S, \circ)$ is a right loop with identity e and a' is also the left inverse for each $a \in S$.

Definition 2.2. [1, Definition 2.9] A right transversal S in a group G to a subgroup H is called a gyrotransversal if

(i) $e \in S$, where e is the identity of the group G,

(ii) $x^{-1} \in S$, for all $x \in S$,

(iii) $h^{-1}xh \in S$, for all $x \in S, h \in H$.

Theorem 2.3. [Representation Theorem for Right Gyrogroups] [1, p. 33] *A right loop $(S, \circ)$ is a right gyrogroup if and only if it is a gyrotransversal to G_S in its group extension $G_S S$.*

Proposition 2.4. [8, Lemma 5.11] *Let S be a gyrotransversal to a subgroup H in a group G and $g : S \longrightarrow H$ be a map such that $g(e) = e$. Then the transversal $S_g = \{g(x)x \mid x \in S\}$ is a gyrotransversal if and only if*
$$g(x^{-1}) = g(x)^{-1}$$
and
$$g(h^{-1}xh) = h^{-1}g(x)h,$$
for all $x \in S$ and $h \in H$.

The map $g : S \longrightarrow H$ such that $g(e) = e$ and satisfying the conditions in the Proposition 2.4 is defined as the deformation map and the gyrotransversal S_g is called the deformed gyrotransversal corresponding to the fixed gyrotransversal S to the subgroup H in a group G. The map g induces a binary operation $\circ_g$ on S defined as, $x \circ_g y = xg(y) \circ y$ for all $x, y \in S$ such that $(S_g, \circ) \simeq (S, \circ_g)$.

Two right transversals are said to be isomorphic if their induced right loop structures are isomorphic. If S is a gyrotransversal to H in G and T is isomorphic transversal, then T is also gyrotransversal to H in G.

Theorem 2.5. [Cauchy-Frobenius Formula] [10, Theorem 3.1.2]
Let a group P acts on a set Y. Then the number of orbits of P on Y is equal to the average number of points left fixed by the elements of G, that is,

$$\textit{number of orbits} = \frac{1}{|P|}\sum_{p\in P} Fix(p),$$

where $Fix(p) = \{y \in Y \mid p \cdot y = y\}$.

3. Gyrotransversals in the Group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$

Let $G = H \ltimes K$ be a group, where H is abelian and K be any group. Let S be a gyrotransversal to the subgroup H in the group G and $g : S \longrightarrow H$ be a deformation map. Then for all $s \in S$ and $h \in H$, we have

$$g(h^{-1}sh) = h^{-1}g(s)h = g(s). \tag{1}$$

Let H acts on the set $S \setminus \{1\}$ by the action defined as $(h, s) \mapsto h^{-1}sh$. Then for any $s \in S \setminus \{1\}$, the orbit of s is given as $\overline{s} = \{h^{-1}sh \mid h \in H\}$. Using the Equation (1), one can easily observe that it is sufficient to find the images of the representatives of the H-orbits on $S \setminus \{1\}$. Let $\{s_1, s_2, \cdots, s_n\}$ be the set of representatives of the H-orbits on $S \setminus \{1\}$. Note that, for all $h, h_j \in H$, $h^{-1}(h_j\overline{s_i})h = h_j(h^{-1}\overline{s_i}h) = h_j\overline{s_i}$. Thus, $h^{-1}sh \in S$, for all $h \in H$ and $s \in S$ holds trivially. Therefore, we only have to check the condition that $S^{-1} = S$ for S to be a gyrotransversal to H in G. Now, we calculate the total number of gyrotransversals to the subgroup H in the group G.

Theorem 3.1. *Let $G = H \ltimes K$ be a group, where H is abelian and K be any group. Then, the total number of gyrotransversals to the subgroup H in the group G is*

$$= \begin{cases} |H|^{\frac{n}{2}}, & \textit{if } n \textit{ is even} \\ |H|^{\frac{n+1}{2}}, & \textit{if } n \textit{ is odd} \end{cases},$$

where n is the number of H-orbits on $S \setminus \{1\}$.

Proof. Note that, any right coset of H in G is of the form Hk, where $k \in K$. Thus using the discussion above the Theorem 3.1, a right transversal S of H in G satisfying the condition $h^{-1}Sh \subseteq S$ is given by

$$S = \bigcup_{i=1}^{n} \{1\} \cup h_{j_i}\overline{s_i},$$

where $h_{j_i} \in H$ and $1 \leqslant i \leqslant n$. Note that, if $h_j s_i \in S$, then $(h_j s_i)^{-1} = s_i^{-1} h_j^{-1} = h_j^{-1}(h_j s_i^{-1} h_j^{-1}) \in h_j^{-1}\overline{s_i^{-1}}$. Therefore, $S^{-1} = S$ if and only if $h_j^{-1}\overline{s_i^{-1}} \subseteq S$ whenever $h_j\overline{s_i} \subseteq S$. Thus, if n is even, then we have to choose only half of the orbits, that is $\frac{n}{2}$ orbits and if n is odd, then we have to choose $\frac{n-1}{2} + 1 = \frac{n+1}{2}$ orbits. Thus, if S is a gyrotransversal to the subgroup H in the group G, then

$$S = \bigcup_{i=1}^{N} \{1\} \cup h_{j_i}\overline{s_i} \cup h_{j_i}^{-1}\overline{s_i^{-1}},$$

where $N = \frac{n}{2}$ or $\frac{n+1}{2}$ accordingly as n is even or odd respectively.

Therefore, the total number of gyrotransversals to the subgroup H in the group G is equal to all the possible choices of the sets $h_{j_i}\overline{s_i}$, for all $h_{j_i} \in H$ and $1 \leqslant i \leqslant N$. Hence, the total number of gyrotransversals are

$$\begin{aligned} &= \sum_{r=0}^{N} {}^{N}C_r \times (|H|-1)^r \\ &= (1 + (|H|-1))^N \\ &= |H|^N. \end{aligned}$$

□

Now, let G denotes the group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$ with the presentation

$$G = \langle a, b \mid a^{p^2} = 1 = b^p, bab^{-1} = a^{1+p} \rangle, \tag{2}$$

where p is an odd prime. From now onwards, G will denote the group as represented in the Equation (2) and H will denote the subgroup $\langle b \rangle$ of G. Then one can easily observe that if S is a gyrotransversal to H in G which is not a group, then $H \simeq G_S$ and $G \simeq G_S S$.

Lemma 3.2. $(b^j a^i)^n = b^{nj} a^{ni - \frac{n(n-1)}{2}pij}$, *for all* n, $0 \leqslant j \leqslant p-1$ *and* $0 \leqslant i \leqslant p^2 - 1$.

Theorem 3.3. *The total number of gyrotransversals to the subgroup H in the group G is equal to p^{p-1}.*

Proof. Note that, Ha^i $(0 \leqslant i \leqslant p^2 - 1)$ are all the right cosets of H in G. Thus, a right transversal of H in G is given by

$$S = \bigcup_{i=1}^{p-1} (\{1\} \cup b^{j_i}\overline{a^i} \cup \{b^{j_i'} a^{pi}\}), \tag{3}$$

where $0 \leqslant j_i, j'_i \leqslant p-1$.

Now, for all $0 \leqslant j, k \leqslant p-1$ and $1 \leqslant i \leqslant p-1$, $b^j(b^k a^i)b^{-j} = b^k(b^j a^i b^{-j}) = b^k a^{i(1+p)^j} = b^k a^{i+pij} = b^k a^{i+pl}$, where $l \equiv ij \pmod p$. Then, under the action $(h,s) \mapsto h^{-1}sh$ of H on the set $S \setminus \{1\}$, the orbit of an element $b^k a^i$ is given as

$$b^k\overline{a^i} = \{b^k a^{i+pl} \mid 0 \leqslant l \leqslant p-1\}.$$

Thus, if S is a gyrotransversal to H in G, then

$$S = \bigcup_{i=1}^{\frac{p-1}{2}} (\{1\} \cup b^{j_i}\overline{a^i} \cup b^{-j_i}\overline{a^{-i}} \cup \{b^{j'_i}a^{pi}\} \cup \{b^{-j'_i}a^{-pi}\}). \tag{4}$$

So, the set of representatives of orbits is

$$\{a, a^2, a^3, \cdots, a^{p-1}\} \cup \{a^p, a^{2p}, \cdots, a^{p(p-1)}\}.$$

Thus total number of orbits is equal to $n = 2(p-1)$ and so, $N = p-1$. Hence, using the Theorem 3.1, the total number of gyrotransversals are p^{p-1}. □

Now, we calculate the isomorphism classes of gyrotransversals to the subgroup H in the group G. We will use the *Cauchy-Frobenius Formula* to find the isomorphism classes of gyrotransversals. Let X denotes the collection of all the gyrotransversals to the subgroup H in the group G. As given in [6], any automorphism $\theta \in Aut_H(G)$ is given by

$$\theta(a) = b^j a^i \text{ and } \theta(b) = b,$$

where $0 \leqslant j \leqslant p-1$ and $i \in \mathbb{Z}_{p^2}$ such that $\gcd(p,i) = 1$. One can easily observe that the map $\theta \in Aut_H(G)$ fixes the elements of H. The restriction map of the map θ to S will give an isomorphism between S and $\theta(S)$. In this way, $Aut_H(G)$ acts naturally on the set X of all gyrotransversals. Note that, the image of any map $\theta \in Aut_H(G)$ depends only on the images of the elements of the subgroup $\langle a \rangle$ of G. So, we will only look at the image of the subgroup $\langle a \rangle$ for any $\theta \in Aut_H(G)$. Therefore, whenever we take $\theta \in Aut_H(G)$, we will define only $\theta(a)$, because $\theta(b) = b$. Now, we find the set $Fix(\theta) = \{S \in X \mid \theta(S) = S\}$, for all $\theta \in Aut_H(G)$.

Lemma 3.4. *Let S be a gyrotransversal to the subgroup H in the group G and θ, θ_l be two automorphisms in $Aut_H(G)$ defined by $\theta(a) = b^j a^k$ and $\theta_l(a) = b^j a^{k+pl}$, for all $1 \leqslant l \leqslant p-1$. Then $\theta(S) = S$ if and only if $\theta_l(S) = S$.*

Proof. Let θ, $\theta_l \in Aut_H(G)$ be defined by $\theta(a) = b^j a^k$ and $\theta_l(a) = b^j a^{k+pl}$, for all $1 \leqslant l \leqslant p-1$. Then using the Lemma 3.2, for all $1 \leqslant i \leqslant p-1$, we have, $\theta(\overline{a^i}) = \overline{\theta(a^i)} = \overline{(b^j a^k)^i} = \overline{b^{ij} a^{ki - \frac{i(i-1)}{2}pjk}} = b^{ij}\overline{a^{ki}}$ and $\theta(a^{pi}) = (b^j a^k)^{pi} = a^{kpi}$. Also, $\theta_l(\overline{a^i}) = \overline{\theta_l(a^i)} = \overline{(b^j a^{k+pl})^i} = \overline{b^{ij} a^{(k+pl)i - \frac{i(i-1)}{2}pj(k+pl)}} = b^{ij}\overline{a^{ki}}$ and $\theta_l(a^{pi}) = (b^j a^{k+pl})^{pi} = a^{kpi}$. Hence, $\theta(S) = S$ if and only if $\theta_l(S) = S$. $\square$

Lemma 3.5. *Let $\theta_i \in Aut_H(G)$ be defined by $\theta_i(a) = a^i$, where $i \equiv 1 \pmod p$. Then $|Fix(\theta_i)| = |X|$, for all $i \in \{1, 1+p, \cdots, 1+p(p-1)\}$.*

Proof. Let $\theta_i(a) = a^i$, where $i \equiv 1 \pmod p$. Then for any $b^j a^{k+pk'} \in b^j\overline{a^k}$, we have $\theta_i(b^j a^{k+pk'}) = b^j a^{k+pk'i} \in b^j\overline{a^k}$, as $i \equiv 1 \pmod p$. Thus, $\theta_i(b^j\overline{a^k}) = b^j\overline{a^k}$ and so, $\theta_i(S) = S$, for all $S \in X$. Hence, using the Lemma 3.4, we get $Fix(\theta_i) = X$, for all $i \in \{1, 1+p, \cdots, 1+p(p-1)\}$. $\square$

Lemma 3.6. *Let $\theta_{ij} \in Aut_H(G)$ be defined by $\theta_{ij}(a) = b^j a^i$, where $i \equiv 1 \pmod p$, $1 \leqslant j \leqslant p-1$. Then $|Fix(\theta_{ij})| = 0$.*

Proof. Let $\theta_{ij}(a) = b^j a^i$, where $i \equiv 1 \pmod p$, $1 \leqslant j \leqslant p-1$ and S be a gyrotransversal to the subgroup H in the group G such that $\theta_{ij}(S) = S$. Then for any $b^k a^l \in b^k\overline{a^l} \subseteq S$, $\theta_{ij}(b^k a^l) = b^k(b^j a^i)^l \in b^{k+jl}\overline{a^{il}} = b^{k+jl}\overline{a^l}$, as $i \equiv 1 \pmod p$. Thus $b^k\overline{a^l}$ and $b^{k+jl}\overline{a^l}$ both are subsets of S which is possible only if $b^{jl} = 1$ which implies p divides jl. Since $\gcd(j,p) = 1$, $l = 0$, which is a contradiction. Therefore, in this case, $Fix(\theta_{ij}) = \emptyset$, for all $i \in \{1, 1+p, \cdots, 1+p(p-1)\}$ and $1 \leqslant j \leqslant p-1$. $\square$

Lemma 3.7. *Let $\theta_{ij} \in Aut_H(G)$ be defined by $\theta_{ij}(a) = b^j a^i$, where $i \not\equiv 1 \pmod p$, $0 \leqslant j \leqslant p-1$ and order of $i \pmod p$ is m. Then*

$$|Fix(\theta_{ij})| = \Gamma_m = \begin{cases} 1, & \textit{if } m \textit{ is even} \\ p^{2\alpha}, & \textit{if } m \textit{ is odd} \end{cases},$$

where $\alpha = \frac{p-1}{2\gcd(\frac{p-1}{2}, m)}$.

Proof. Let $\theta_{ij}(a) = b^j a^i$, where $i \not\equiv 1 \pmod p$, $0 \leqslant j \leqslant p-1$ and the order of $i \pmod p$ be m. Let S be a gyrotransversal to H in G such that $\theta_{ij}(S) = S$. Using the Equation (4), elements of the set S are either of the form $b^k a^{pl}$ or the elements in the sets $b^k\overline{a^l}$, where $1 \leqslant l \leqslant p-1$ and $0 \leqslant k \leqslant p-1$. Now, we find the images of these elements under the map θ_{ij}. Then,

$\theta_{ij}(b^k a^{pl}) = b^k a^{ipl}$ and $\theta_{ij}(b^k a^{ipl}) = b^k a^{i^2 pl}$ and so on. Similarly, if $b^k\overline{a^l} \subseteq S$, then $\theta_{ij}(b^k\overline{a^l}) = b^{k+jl}\overline{a^{il}} \subseteq S$. Again, $\theta_{ij}(b^{k+jl}\overline{a^{il}}) = b^{k+jl+ijl}\overline{a^{i^2 l}} \subseteq S$ and so on. Thus, we get the sets

$$T_{(k,l)} = b^k\overline{a^l} \cup \bigcup_{r=1}^{m-1} b^{k+jl(1+i+\cdots+i^{r-1})}\overline{a^{i^r l}} \subseteq S \tag{5}$$

and

$$T'_{(k,l)} = \{b^k a^{pl}, b^k a^{ipl}, b^k a^{i^2 pl}, \cdots, b^k a^{i^{m-1}pl}\} \subseteq S$$

such that S is the union of all such sets $T_{(k,l)}$ and $T'_{(k,l)}$, $\theta_{ij}(T_{(k,l)}) = T_{(k,l)}$ and $\theta_{ij}(T'_{(k,l)}) = T'_{(k,l)}$. Now, we find all such sets $T_{(k,l)}$ and $T'_{(k,l)}$. Then all the possible combinations of these sets will give all the gyrotransversals that are fixed by the map θ_{ij}. For this, we consider two cases, first, when m is even and second, when m is odd.

Case(i). Let m be even. Then note that, $i^{\frac{m}{2}} \equiv -1 \pmod{p}$ and $\frac{m}{2} + u \equiv -(\frac{m}{2} - u) \pmod{m}$, for all u. Now, $b^{k+jl(1+i+\cdots+i^{\frac{m}{2}-1})}\overline{a^{i^{\frac{m}{2}} l}} = b^{k+jl(1+i+\cdots+i^{\frac{m}{2}-1})}\overline{a^{-l}}$ and $(b^k\overline{a^l})^{-1} = b^{-k}\overline{a^{-l}}$. Therefore, using $S^{-1} = S$, we have

$$\begin{aligned} & k + jl(1 + i + i^2 + \cdots + i^{\frac{m}{2}-1}) \equiv -k \pmod{p} \\ \Longrightarrow\ & 2k + jl\left(\frac{i^{\frac{m}{2}} - 1}{i-1}\right) \equiv\ 0 \pmod{p} \\ \Longrightarrow\ & 2k + jl\left(\frac{-2}{i-1}\right) \equiv\ 0 \pmod{p} \\ \Longrightarrow\ & k(i-1) \equiv\ jl \pmod{p}. \end{aligned}$$

Since for given l and j, k is unique that satisfies the above congruence relation, there is only one set $T_{(k,l)}$ such that $\theta_{ij}(T_{(k,l)}) = T_{(k,l)}$ and the number of such sets $T_{(k,l)}$ is equal to $\alpha = \frac{p-1}{2\gcd(\frac{p-1}{2},m)}$. By the similar argument, we get $\theta_{ij}(T'_{(k,l)}) = T'_{(k,l)}$ if and only if $k = 0$ and the total number of such sets $T'_{(k,l)}$ is equal to $\alpha = \frac{p-1}{2\gcd(\frac{p-1}{2},m)}$. Since k is unique, in this case, $Fix(\theta_{ij}) = \{S\}$.

Case(ii). Let m be odd. Note that, if $b^{k'}\overline{a^{l'}}$ and $b^{-k'}\overline{a^{-l'}}$ both lies in the set $T_{(k,l)}$, then using $\theta_{ij}(T_{(k,l)}) = T_{(k,l)}$, we get $|T_{(k,l)}| = m =$ even, which is a contradiction. Therefore, only one of $b^{k'}\overline{a^{l'}}$ or $b^{-k'}\overline{a^{-l'}}$ lies in the set $T_{(k,l)}$. Thus, we only have to focus on half of such sets $T_{(k,l)}$, that is $\frac{p-1}{2}$

sets, as other $\frac{p-1}{2}$ sets are the inverses of these sets. Now, the number of such sets $T_{(k,l)}$ is $\alpha = \frac{p-1}{2\gcd(\frac{p-1}{2},m)}$. By the similar argument, we have the number of sets $T'_{(k,l)}$ is $\alpha = \frac{p-1}{2\gcd(\frac{p-1}{2},m)}$. Let us denote such sets as $T_{(k,l)_r}$ and $T'_{(k,l)_s}$, where $1 \leqslant r, s \leqslant \alpha$. Then, such a gyrotransversal S is given by

$$S = \bigcup_{1 \leqslant r,s \leqslant \alpha} (\{1\} \cup T_{(k,l)_r} \cup {T_{(k,l)_r}}^{-1} \cup T'_{(k,l)_s} \cup {T'_{(k,l)_s}}^{-1}).$$

Also, there are p choices of k for each set $T_{(k,l)_r}$ and $T'_{(k,l)_s}$. Hence, the total number of gyrotransversals fixed by the map θ_{ij} is equal to $p^\alpha \times p^\alpha = p^{2\alpha}$. □

Theorem 3.8. *The number of isomorphism classes of gyrotransversals to the subgroup H in the group G is equal to*

$$\frac{1}{p-1}\left(p^{p-2} + \sum_{m \in d(p-1)} \phi(m)\Gamma_m\right), \tag{6}$$

where Γ_m is defined as in the Lemma 3.7.

Proof. Using the Lemmas 3.4, 3.5, 3.6, 3.7 and the *Cauchy - Frobenius Formula*, we have, the number of isomorphism classes of gyrotransversals is equal to the number of orbits, that is,

$$\begin{aligned}
&= \frac{1}{|Aut_H(G)|} \sum_{\theta \in Aut_H(G)} Fix(\theta) \\
&= \frac{1}{p^2(p-1)} \left(p \times p^{p-1} + 0 + p^2 \times \sum_{\substack{1 \neq i \in U(p) \\ o(i)=m}} \Gamma_m \right) \\
&= \frac{1}{p-1}\left(p^{p-2} + \sum_{m \in d(p-1)} \phi(m)\Gamma_m\right).
\end{aligned}$$

□

Corollary 3.9. *The lower bound for the number of right gyrogroups of order p^2 is given by the number in* (6).

Proof. Using the Theorems 2.3 and 3.8, the result holds. □

Next, we will find the lower bound of non-isomorphic gyrotransversals such that the corresponding right gyrogroup is of nilpotency class 2. Note that, the center of the group G, $Z(G) = \langle a^p \rangle$. Now, we will find the deformations of the gyrotransversals such that $g(Z(G)) = 1$, that is, a gyrotransversal of the form given below,

$$T = \overline{a^p} \cup \bigcup_{i=1}^{p-1} b^j \overline{a^i}.$$

Theorem 3.10. *Let T be any gyrotransversal to the subgroup H in the group G such that $g(Z(G)) = 1$. Then T is of nilpotency class 2.*

Proof. Note that $S = \{1, a, a^2, \cdots, a^{p^2-1}\}$ is a gyrotransversal to the subgroup H in the group G. Let $g : S \longrightarrow H$ be a deformation map such that $g(Z(G)) = 1$. Then, a corresponding gyrotransversal T is given as,

$$T = Z(G) \cup \bigcup_{i=1}^{\frac{p-1}{2}} (b^j \overline{a^i} \cup b^{-j} \overline{a^{-i}}).$$

Since $Z(G)$ defines a unique maximal central congruence on the group G, $Z(G)$ also defines a unique maximal central congruence on the right loop T (see [2, Definition 2.3, p. 5123]). Therefore, $Z(G)$ is the center of the right loop T and $T/Z(G)$ is a right loop of order p. The map $\eta : T \longrightarrow T/Z(G)$ is a natural right loop homomorphism defined by $\eta(x) = Z(G) \circ x$. Now, the map η induces a group homomorphism $\overline{\eta} : G_T T \longrightarrow G_{T/Z(G)} T/Z(G)$. Then the restriction map $\overline{\eta}|_{G_T} : G_T \longrightarrow G_{T/Z(G)}$ is an onto group homomorphism. Since $|G_T| = p$, $|G_{T/Z(G)}| = 1$ or p. If $|G_{T/Z(G)}| = p$, then $|G_{T/Z(G)} T/Z(G)| = p^2$ and so, the group $G_{T/Z(G)} T/Z(G)$ is an abelian group. Therefore, the core of the group $G_{T/Z(G)}$ in the group $G_{T/Z(G)} T/Z(G)$ is $G_{T/Z(G)} \neq \{1\}$, which is a contradiction to the fact that $Core_{G_T T}(G_T)$ is always trivial (see [9, Proposition 1.8, p. 2683]). Thus, $|G_{T/Z(G)}| = 1$ and so, $T/Z(G)$ is a group of order p. Hence, we have a central extension

$$0 \longrightarrow Z(G) \xrightarrow{i} T \xrightarrow{\eta} T/Z(G) \longrightarrow 1 .$$

Since, $T/Z(G)$ is an abelian group, T is a nilpotent right loop of class 2. Thus, T is a gyrotransversal of nilpotency class 2. $\square$

Now, we find the total number of such gyrotransversals to the subgroup H in the group G.

Theorem 3.11. *The total number of gyrotransversals in the group G to the subgroup H such that $g(Z(G)) = 1$ is equal to $p^{\frac{p-1}{2}}$.*

Proof. The proof is similar to the proof of the Theorem 3.3. □

Theorem 3.12. *The number of isomorphism classes of gyrotransversals to the subgroup H in the group G such that $g(Z(G)) = 1$ is equal to*

$$\frac{1}{p-1}\left(p^{\frac{p-3}{2}} + \sum_{m \in d(p-1)} \phi(m)\Gamma'_m\right), \tag{7}$$

where

$$\Gamma'_m = \begin{cases} 1, & \text{if } m \text{ is even} \\ p^{\alpha}, & \text{if } m \text{ is odd} \end{cases}$$

Proof. The proof is similar to the proof of the Theorem 3.8. □

Corollary 3.13. *The lower bound for the number of right gyrogroups of order p^2 having the nilpotency class 2 is given by the number in* (7).

Proof. Using the Theorems 2.3, 3.12 and 3.10, the proof follows immediately. □

3.1 Examples

Example 1. Consider the group $G = \mathbb{Z}_3 \ltimes \mathbb{Z}_9 = \langle a, b \mid a^9 = 1 = b^3, bab^{-1} = a^4\rangle$ and the subgroup $H = \langle b\rangle$ of order 3. Then there are $3^{3-1} = 9$ gyrotransversals to the subgroup H in the group G listed as follows,

$S_1 = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$,
$S_2 = \{1, a, a^2, ba^3, a^4, a^5, b^2a^6, a^7, a^8\}$,
$S_3 = \{1, a, a^2, b^2a^3, a^4, a^5, ba^6, a^7, a^8\}$,
$S_4 = \{1, ba, b^2a^2, a^3, ba^4, b^2a^5, a^6, ba^7, b^2a^8\}$,
$S_5 = \{1, ba, b^2a^2, ba^3, ba^4, b^2a^5, b^2a^6, ba^7, b^2a^8\}$,
$S_6 = \{1, ba, b^2a^2, b^2a^3, ba^4, b^2a^5, ba^6, ba^7, b^2a^8\}$,
$S_7 = \{1, b^2a, ba^2, a^3, b^2a^4, ba^5, a^6, b^2a^7, ba^8\}$,
$S_8 = \{1, b^2a, ba^2, ba^3, b^2a^4, ba^5, b^2a^6, b^2a^7, ba^8\}$,
$S_9 = \{1, b^2a, ba^2, b^2a^3, b^2a^4, ba^5, ba^6, b^2a^7, ba^8\}$.

Now, any map $\theta \in Aut_H(G)$ is given by

$$\theta(a) = b^j a^i \text{ and } \theta(b) = b,$$

where $j = 0, 1, 2$ and $i \in U(9)$. So, $|Aut_H(G)| = 18$. Now, one can easily check that the maps $\theta_i(a) = a^i$, where $i \in \{1, 4, 7\}$ fixes all the gyrotransversals. Therefore, $|Fix(\theta_i)| = 9$, for all $i \in \{1, 4, 7\}$. The maps $\theta_l(a) = a^l$, where $l \in \{2, 5, 8\}$ fixes only one gyrotransversal S_1. So, $|Fix(\theta_l)| = 1$, for all $l \in \{2, 5, 8\}$. The maps $\theta_{ij}(a) = b^j a^i$, where $i \in \{1, 4, 7\}$, $j \in \{1, 2\}$ do not fix any gyrotransversal. Therefore, $|Fix(\theta_{ij})| = 0$. At last, the maps $\theta_l(a) = ba^l$, where $l \in \{2, 5, 8\}$ fixes only one gyrotransversal S_4 and the maps $\theta_l(a) = b^2 a^l$, where $l \in \{2, 5, 8\}$ fixes only one gyrotransversal S_7. Thus,

$$\sum_{\theta \in Aut_H(G)} Fix(\theta) = (3 \times 9) + (3 \times 1) + 0 + (3 \times 1) + (3 \times 1) = 36.$$

Hence, using the *Cauchy - Frobenius Formula*, we get, the number of orbits is equal to $\frac{36}{18} = 2$.

On the other hand, one can easily check that under the map defined as $\theta(a^i) = (ba)^i$, $\theta(b) = b$, $S_1 \simeq S_4 \simeq S_7$, $S_2 \simeq S_5 \simeq S_8$ and $S_3 \simeq S_6 \simeq S_9$. Also, under the map defined as $\theta(a^i) = a^{2i}$, $\theta(b) = b$, $S_2 \simeq S_3$. Hence, we get two classes of isomorphism of gyrotransversals given as $\{S_1, S_4, S_7\}$ and $\{S_2, S_3, S_5, S_6, S_8, S_9\}$.

Also, the number of gyrotransversals of H in the group G such that $g(Z(G)) = 1$ is equal to 3, namely S_1, S_4 and S_7 which are isomorphic to each other as shown above, where $Z(G) = \{1, a^3, a^6\}$. Hence, the number of isomorphism classes of gyrotransversals of order 9 having nilpotency class 2 is equal to 1. Also, by the Theorem 3.12, number of orbits is equal to $\frac{3+1+2\times 1}{6} = 1$.

Example 2. Using the similar process as in the above example, we have checked that the number of isomorphism classes of gyrotransversals in the group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$ to the subgroup $\mathbb{Z}_p$, for $p = 5, 7, 11$ and 13, is equal to 32, 2818, 235794818 and 149346704264 respectively. Also, the number of isomorphism classes of gyrotransversals of order 25, 49, 121 and 169 having nilpotency class 2 is equal to 2, 11, 1469 and 30970 respectively.

4. Discussion on the lower bound of gyrotransversals

Let $Sym(n)$ denotes the symmetric group of degree n. In [5], it is observed that all non-isomorphic groups of order n can be realized as right transversals in $Sym(n)$ to the subgroup $Sym(n-1)$ of $Sym(n)$. In [9, Theorem 3.7, p. 2693], it is observed that all non-isomorphic right loops(in particular all non-isomorphic loops) of order n can be realized as right transversals in $Sym(n)$ to the subgroup $Syn(n-1)$. But, it is not true for the right gyrogroups. In [8, Corollary 5.13, p. 3570] it is observed that there is a unique gyrotransversal in $Sym(n)$ to the subgroup $Sym(n-1)$. This prompts us to pose the following problem.

Problem: *Classify the pair (G, H) of group G and its subgroup H with index $[G : H] = n$, which contains all non-isomorphic right gyrogroups of order n as gyrotransversals.*

By Theorem 2.3, a right loop $(S, \circ)$ is a right gyrogroup if and only if it is a gyrotransversal in $G_S S$ to the subgroup G_S. By [7, Proposition 1.9, p. 647], the core, $Core_{G_S S}(G_S)$ of G_S in $G_S S$ is trivial. This implies that $G_S S$ is realized as a subgroup of $Sym(m)$, where $|S| = m$. To get the information of right gyrogroups of order 9, we have searched various possible subgroups G of $Sym(9)$ and a subgroup H of G with $[G : H] = 9$. Using GAP [12], we found the isomorphism classes of gyrotransversals in G to the subgroup H, for various possible cases. We found that there is a subgroup G of $Sym(9)$ isomorphic to $\mathbb{Z}_6 \times Sym(3)$ and $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ such that the number of isomorphism classes of gyrotransversals in $G \simeq \mathbb{Z}_6 \times Sym(3)$ to $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is 75. We are not sure that these are all non-isomorphic right gyrogroups of order 9. But there are at least 75 non-isomorphic right gyrogroups of order 9.

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