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A decomposition of Γ-bands

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Abstract. In this paper we define the notion of the rectangular Γ-band, and give equivalent conditions on a Γ-semigroup to be a rectangular Γ-band. Moreover, we show that every Γ-band is a Γ-semilattice of rectangular Γ-bands.

1. Introduction and motivation

A nonempty set together with an associative binary operation is called a semigroup. The notion of Γ-semigroup is a generalization of the notion of a semigroup. Let Γ be a nonempty set. Using the terminology of [4], we say that a nonempty set S is a Γ-semigroup if there exists a mapping of $S\times\Gamma\times S$ into S written as $(a,\alpha,b)\mapsto a\alpha b$ satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$. One of the central topics in the theory of semigroups is the decomposition of semigroups into different types of semigroups. The readers are referred to the books [1, 2, 3]. The main result of this area is that every semigroup is a semilattice of semilattice indecomposable semigroups. Especially, every band is a semilattice of rectangular bands. In this paper we extend this last result to Γ-bands. In Section 3, we define the notion of the rectangular Γ-band and give equivalent conditions for a Γ-semigroup to be a rectangular Γ-band. In Section 4, we define a binary relation η_S on a Γ-band S, and show that η_S is the least Γ-congruence on S such that the factor Γ-semigroup S/η_S is a Γ-semilattice. We also show that the η_S-classes of S are rectangular Γ-bands. Thus every Γ-band is a Γ-semilattice of rectangular Γ-bands.

2. Preliminaires

Let S be a Γ-semigroup and $\alpha\in\Gamma$ be an arbitrary element. We say that an element e of S is an *α-idempotent* if $e\alpha e = e$. An element e of a Γ-

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semigroup S is called a Γ-*idempotent* if e is an α-idempotent for all $\alpha \in \Gamma$. A Γ-semigroup S will be called a Γ-*band* if every element of S is a Γ-idempotent.

A Γ-semigroup S in which $a\alpha b = b\alpha a$ is satisfied for every $a, b \in S$ and every $\alpha \in \Gamma$ is called a *commutative* Γ-*semigroup*. A commutative Γ-band is called a Γ-*semilattice*. By a *nowhere commutative* Γ-*semigroup* we mean a Γ-semigroup in which $a\alpha b = b\alpha a$ implies $a = b$ for every $a, b \in S$ and $\alpha \in \Gamma$.

A mapping φ of a Γ-semigroup S_1 into a Γ-semigroup S_2 is said to be a Γ-*homomorphism* if $\varphi(a\alpha b) = \varphi(a)\alpha\varphi(b)$ is satisfied for every $a, b \in S_1$ and every $\alpha \in \Gamma$. A bijective Γ-homomorphism is called a Γ-*isomorphism*.

An equivalence relation σ on a Γ-semigroup is called a Γ-*congruence* on S if $a\sigma c$ and $b\sigma d$ imply $a\alpha b\sigma c\alpha d$ for every $a, b, c, d \in S$ and $\alpha \in \Gamma$. An equivalence relation σ on a Γ-semigroup is called a *left* Γ-*congruence* on S if $a\sigma b$ implies $c\alpha a\sigma c\alpha b$ for every $a, b, c \in S$ and $\alpha \in \Gamma$. The notion of *right* Γ-*congruence* is the dual of the notion of a left Γ-congruence. It is easy to see that an equivalence relation of a Γ-semigroup S is a Γ-congruence if and only if it is a left Γ-congruence on S and a right Γ-congruence on S.

If σ is a Γ-congruence on a Γ-semigroup S, then the factor set S/σ is also a Γ-semigroup: for arbitrary $\alpha \in \Gamma$ and arbitrary σ-classes A and B of S, $A\alpha B = C$, where C is the σ-class of S which contains the elements of $A\alpha B$. This Γ-semigroup is called the *factor* Γ-*semigroup of* S (modulo σ).

We say that σ is a Γ-*semilattice* Γ-*congruence* on a Γ-semigroup S if σ is a Γ-congruence on S such that the factor Γ-semigroup S/σ is a Γ-semilattice.

Let S be a Γ-semigroup, and let S_i $(i \in I)$ be pairwise disjoint Γ-subsemigroups of S such that $S = \cup_{i\in I} S_i$. If the equivalence relation on S whose classes are the subsets S_i $(i \in I)$ is a semilattice Γ-congruence on S, then we say that S is a Γ-*semilattice of* Γ-*semigroups* S_i $(i \in I)$.

3. Rectangular Γ-bands

Definition 3.1. A Γ-band S will said to be a *rectangular* Γ-*band* if it satisfies the identity $a\alpha b\alpha a = a$, that is, $a\alpha b\alpha a = a$ is satisfied for all $a, b \in S$ and all $\alpha \in \Gamma$.

The next theorem gives equivalent conditions for a Γ-semigroup to be a rectangular Γ-band.

Theorem 3.2. *The following conditions on a* Γ-*semigroup* S *are equivalent.*

(1) *S is a rectangular Γ-band.*

(2) *S is a Γ-band satisfying the identity $a\alpha b\beta a = a$.*

(3) *S is a Γ-band satisfying the identity $a\alpha b\beta c = a\gamma c$.*

(4) *S is a Γ-band satisfying the identity $a\alpha b\alpha c = a\alpha c$.*

(5) *S is Γ-isomorphic to the direct product of a left zero Γ-semigroup L and a right zero Γ-semigroup R.*

(6) *S is a nowhere commutative Γ-semigroup.*

Proof. (1) $\Rightarrow$ (2): Assume that S is a rectangular Γ-band. Let $a, b \in S$ and $\alpha, \beta \in \Gamma$ be arbitrary elements. Then

$$a\alpha b\beta a = a\alpha b\beta(a\alpha a) = a\alpha(b\beta a)\alpha a = a.$$

(2) $\Rightarrow$ (3): Assume that S is a Γ-band satisfying the identity $a\alpha b\beta a = a$. Let $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$ be arbitrary elements. Then

$$a\alpha b\beta c = a\alpha b\beta(c\gamma a\gamma c) = (a\alpha(b\beta c)\gamma a)\gamma c = a\gamma c.$$

Thus (3) is satisfied.

(3) $\Rightarrow$ (4): It is obvious.

(4) $\Rightarrow$ (5): Assume that S is a Γ-band satisfying the identity $a\alpha b\alpha c = a\alpha c$. Let $e \in S$ and $\alpha \in \Gamma$ be arbitrary elements. Let

$$L = S\alpha e \quad \text{and} \quad R = e\alpha S.$$

For arbitrary $a, b \in S$ and $\beta \in \Gamma$, we have

$$(a\alpha e)\beta(b\alpha e) = a\alpha(e\beta b)\alpha e = a\alpha e$$

which implies that L is a left zero Γ-semigroup. Similarly, R is a right zero Γ-semigroup.

We show that S is Γ-isomorphic to the direct product $L \times R$. Let φ be a mapping of $L \times R$ into S defined by the following way: for an element $(s\alpha e, e\alpha t) \in L \times R$, let

$$\varphi(s\alpha e, e\alpha t) = s\alpha e\alpha t.$$

For every $s \in S$, we have

$$s = s\alpha s = s\alpha e\alpha s = \varphi(s\alpha e, e\alpha s),$$

and hence φ is surjective.

We show that φ is injective. Assume

$$\varphi(s\alpha e, e\alpha t) = \varphi(u\alpha e, e\alpha v)$$

for some $s, t, u, v \in S$. Then

$$s\alpha e\alpha t = u\alpha e\alpha v.$$

Applying this equation, we get

$$s\alpha e = s\alpha(e\alpha t\alpha e) = (s\alpha e\alpha t)\alpha e = (u\alpha e\alpha v)\alpha e = u\alpha(e\alpha v\alpha e) = u\alpha e$$

and

$$e\alpha t = (e\alpha s\alpha e)\alpha t = e\alpha(s\alpha e\alpha t) = e\alpha(u\alpha e\alpha v) = (e\alpha u\alpha e)\alpha v = e\alpha v.$$

Thus

$$(s\alpha e, e\alpha t) = (u\alpha e, e\alpha v).$$

Hence φ is injective.

It remains to show that φ is a Γ-homomorphism. Let $(s\alpha e, e\alpha t)$ and $(u\alpha e, e\alpha v)$ be arbitrary elements of $L \times R$. Then, for an arbitrary $\beta \in \Gamma$, we have

$$\varphi((s\alpha e, e\alpha t)\beta(u\alpha e, e\alpha v)) = \varphi((s\alpha e)\beta(u\alpha e), (e\alpha t)\beta(e\alpha v)) =$$

$$= \varphi(s\alpha e, e\alpha v) = s\alpha e\alpha v = s\alpha(e\alpha(t\beta u)\alpha e)\alpha v =$$

$$= (s\alpha e\alpha t)\beta(u\alpha e\alpha v) = \varphi(s\alpha e, e\alpha t)\beta\varphi(u\alpha e, e\alpha v).$$

Thus φ is a Γ-homomorphism. Consequently S is Γ-isomorphic to $L \times R$. Hence (5) is satisfied.

(5) $\Rightarrow$ (6): Assume that the Γ-semigroup S is Γ-isomorphic to the direct product of a left zero Γ-semigroup L and a right zero Γ-semigroup R. If

$$(a_1, b_1)\alpha(a_2, b_2) = (a_2, b_2)\alpha(a_1, b_1)$$

for some $a_1, a_2 \in L$, $b_1, b_2 \in R$ and $\alpha \in \Gamma$, then

$$(a_1, b_2) = (a_1\alpha a_2, b_1\alpha b_2) = (a_1, b_1)\alpha(a_2, b_2) =$$

$$= (a_2, b_2)\alpha(a_1, b_1) = (a_2\alpha a_1, b_2\alpha b_1) = (a_2, b_1)$$

from which we get

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2,$$

that is,

$$(a_1, b_1) = (a_2, b_2).$$

Thus S is nowhere commutative.

(6) ⇒ (1): Assume that S is a nowhere commutative Γ-semigroup. Let $a \in S$ and $\alpha \in \Gamma$ be arbitrary elements. Since

$$a\alpha(a\alpha a) = a\alpha(a\alpha a),$$

that is, a and $a\alpha a$ commute with each other, we get

$$a\alpha a = a.$$

Thus every element of S is a Γ-idempotent, that is, S is a Γ-band. For arbitrary $a, b \in S$ and $\alpha \in \Gamma$, we have

$$a\alpha(a\alpha b\alpha a) = (a\alpha a)\alpha(b\alpha a) = a\alpha b\alpha a = (a\alpha b)\alpha(a\alpha a) = (a\alpha b\alpha a)\alpha a,$$

that is, a and $a\alpha b\alpha a$ commute with each other. Thus

$$a\alpha b\alpha a = a.$$

Consequently S is a rectangular Γ-band, and hence (1) is satisfied. □

An element e of a semigroup S is called an *idempotent element* if $e^2 = e$. A semigroup S is called a *band* if every element of S is an idempotent element. A band satisfying the identity $aba = a$ is called a *rectangular band.*

Theorem 3.3. *Let S and Γ be arbitrary non-empty sets. Then S is a rectangular Γ-band if and only if there is a binary operation $\star$ on S such that $(S; \star)$ is a rectangular band.*

Proof. Assume that S is a rectangular Γ-band. By Theorem 3.2 S satisfies the identity $a\alpha b\alpha c = a\alpha c$. Let $a, b \in S$ and $\alpha, \beta \in \Gamma$ be arbitrary elements. Then

$$a\alpha b = (a\beta a)\alpha(b\beta b) = a\beta(a\alpha b)\beta b = a\beta b.$$

Thus

$$|a\Gamma b| = 1.$$

Let $\star$ be the binary operation on S defined by the following way: for arbitrary $a, b \in S$, let

$$a \star b = a\Gamma b.$$

It is a matter of checking to see that $(S; \star)$ is a rectangular band.

Conversely, assume that there is a binary operation $\star$ on S such that $(S; \star)$ is a rectangular band. For every $a, b \in S$ and $\alpha \in \Gamma$, let

$$a\alpha b = a \star b.$$

It is easy to see that S becomes a rectangular Γ-band. □

4. Γ-bands

Theorem 4.1. *On an arbitrary* Γ*-band* S*,*

$$\eta_S = \{(a, b) \in S \times S : (\forall \alpha \in \Gamma)\ a\alpha b\alpha a = a,\ b\alpha a\alpha b = b\}$$

is the least Γ*-semilattice* Γ*-congruence on* S *such that the* η_S*-classes of* S *are rectangular* Γ*-bands.*

Proof. Let S be a Γ-band. As every element a of S is a Γ-idempotent, we have

$$a\alpha a\alpha a = a\alpha a = a$$

for every $\alpha \in \Gamma$, and hence $a\ \eta_S\ a$. Thus η_S is reflexive. It is obvious that η_S is symmetric. To show that η_S is transitive, let $a, b, c \in S$ be arbitrary elements such that $a\ \eta_S\ b$ and $b\ \eta_S\ c$. Then, for every $\alpha \in \Gamma$,

$$\begin{aligned} a = a\alpha b\alpha a = a\alpha(b\alpha c\alpha b)\alpha a = (a\alpha b)\alpha(c\alpha b\alpha a) = \\ = (a\alpha b)\alpha(c\alpha b\alpha a)\alpha(c\alpha b\alpha a) = a\alpha(b\alpha c\alpha b)\alpha a\alpha(c\alpha b\alpha a) = \\ = (a\alpha b\alpha a)\alpha(c\alpha b\alpha a) = a\alpha c\alpha b\alpha a \end{aligned} \tag{1}$$

and

$$\begin{aligned} a = a\alpha b\alpha a = a\alpha(b\alpha c\alpha b)\alpha a = (a\alpha b\alpha c)\alpha(b\alpha a) = \\ = (a\alpha b\alpha c)\alpha(a\alpha b\alpha c)\alpha(b\alpha a) = (a\alpha b\alpha c)\alpha a\alpha(b\alpha c\alpha b)\alpha a = \\ = (a\alpha b\alpha c)\alpha(a\alpha b\alpha a) = a\alpha b\alpha c\alpha a. \end{aligned} \tag{2}$$

From (1) and (2), we get

$$\begin{aligned} a = a\alpha a = (a\alpha c\alpha b\alpha a)\alpha(a\alpha b\alpha c\alpha a) = (a\alpha c)\alpha(b\alpha a\alpha a\alpha a\alpha b)\alpha(c\alpha a) = \\ = (a\alpha c)\alpha(b\alpha a\alpha b)\alpha(c\alpha a) = (a\alpha c)\alpha b\alpha(c\alpha a) = \\ = a\alpha(c\alpha b\alpha c)\alpha a = a\alpha c\alpha a. \end{aligned} \tag{3}$$

By symmetry, we have

$$c = c\alpha a\alpha c. \tag{4}$$

Equations (3) and (4) together imply that, for every $\alpha \in \Gamma$,

$$a = a\alpha c\alpha a \quad \text{and} \quad c = c\alpha a\alpha c$$

which means that $a\ \eta_S\ c$. Thus η_S is transitive.

We show that η_S is a right Γ-congruence on S. Assume $a\ \eta_S\ b$ for elements $a, b \in S$. Let $c \in S$ and $\alpha, \xi, \kappa \in \Gamma$ be arbitrary elements. Then

$$\begin{aligned}(a\xi c)\kappa(a\alpha b\xi c)\kappa(a\xi c) = (a\xi c)\kappa(a\alpha b\xi c)\kappa((a\alpha b\alpha a)\xi c) = \\ a\xi(c\kappa a\alpha b)\xi(c\kappa a\alpha b)\alpha(a\xi c) = a\xi(c\kappa a\alpha b)\alpha(a\xi c) = \\ = (a\xi c)\kappa(a\alpha b\alpha a)\xi c = (a\xi c)\kappa(a\xi c) = (a\xi c).\end{aligned} \tag{5}$$

Moreover,

$$\begin{aligned}(a\alpha b\xi c)\kappa(a\xi c)\kappa(a\alpha b\xi c) = (a\alpha b)\xi((c\kappa a)\xi(c\kappa a))\alpha(b\xi c) = \\ = (a\alpha b)\xi(c\kappa a)\alpha(b\xi c) = (a\alpha b\xi c)\kappa(a\alpha b\xi c) = a\alpha b\xi c.\end{aligned} \tag{6}$$

Equations (5) and (6) together imply

$$(a\xi c)\ \eta_S\ (a\alpha b\xi c). \tag{7}$$

Since

$$\begin{aligned}(b\xi c)\kappa(a\alpha b\xi c)\kappa(b\xi c) = (b\xi c)\kappa(a\alpha b\xi c)\kappa((b\alpha a\alpha b)\xi c) = \\ ((b\alpha a\alpha b)\xi c)\kappa(a\alpha b\xi c)\kappa(b\xi c) = \\ = b\alpha(a\alpha b\xi c)\kappa(a\alpha b\xi c)\kappa(b\xi c) = \\ = (b\alpha b\alpha a)\xi c\kappa(b\xi c) = (b\xi c)\kappa(b\xi c) = b\xi c\end{aligned} \tag{8}$$

and

$$\begin{aligned}(a\alpha b\xi c)\kappa(b\xi c)\kappa(a\alpha b\xi c) = a\alpha(b\xi c)\kappa(b\xi c)\kappa(a\alpha b\xi c) = \\ a\alpha(b\xi c)\kappa(a\alpha b\xi c) = (a\alpha b\xi c)\kappa(a\alpha b\xi c) = a\alpha b\xi c,\end{aligned} \tag{9}$$

equations (8) and (9) together imply

$$(b\xi c)\ \eta_S\ (a\alpha b\xi c). \tag{10}$$

By (7) and (10), we have

$$(a\xi c)\ \eta_S\ (a\alpha b\xi c)\ \eta_S\ (b\xi c).$$

Since η_S is transitive, we get

$$(a\xi c)\ \eta_S\ (b\xi c).$$

Consequently η_S is a right Γ-congruence on S . We can prove in a similar way that η_S is a left Γ-congruence on S. Thus η_S is a Γ-congruence on S.

For arbitrary $a, b \in S$ and $\alpha, \beta \in \Gamma$,

$$\begin{aligned}(a\alpha b)\beta(b\alpha a)\beta(a\alpha b) = a\alpha(b\beta b)\alpha(a\beta a)\alpha b = \\ = (a\alpha b)\alpha(a\alpha b) = a\alpha b.\end{aligned} \tag{11}$$

We can prove in a similar way that

$$(b\alpha a)\beta(a\alpha b)\beta(b\alpha a) = b\alpha a. \tag{12}$$

By (11) and (12), we get

$$(a\alpha b)\ \eta_S\ (b\alpha a),$$

from which it follows that the factor Γ-semigroup S/η_S is commutative.

It is clear that every η_S-class of S is α-idempotent in S/η_S for every $\alpha \in \Gamma$. Thus S/η_S is a Γ-semilattice. In other words, η_S is a Γ-semilattice Γ-congruence on S.

We show that η_S is the least Γ-semilattice Γ-congruence on S. Let σ be a Γ-semilattice Γ-congruence on S. Let a and b be elements of S such that $a\ \eta_S\ b$. Let $\alpha \in \Gamma$ be an arbitrary element. Then

$$a = a\alpha b\alpha a\ \sigma\ a\alpha(b\alpha b)\alpha a\ \sigma\ (a\alpha a)\alpha(b\alpha b)\ \sigma\ a\alpha(b\alpha b)\ \sigma\ b\alpha a\alpha b = b.$$

Thus

$$a\ \sigma\ b,$$

and hence

$$\eta_S \subseteq \sigma.$$

Consequently η_S is the least Γ-semilattice Γ-congruence on S.

It remains to show that every η_S-class of S is a rectangular Γ-band. Let A be an arbitrary η_S-class of S. As every element of S is Γ-idempotent, A is a Γ-band. Let $a, b \in A$, and let $\alpha \in \Gamma$ be arbitrary elements. Then

$$a\ \eta_S\ b$$

which implies

$$a\alpha b\alpha a = a.$$

Thus the Γ-band A satisfies the identity

$$a\alpha b\alpha a = a.$$

By Theorem 1, A is a rectangular Γ-band. □

Corollary 4.2. *Every* Γ*-band is a* Γ*-semilattice of rectangular* Γ*-bands.*

Proof. By Theorem 4.1, it is obvious. □

Corollary 4.3. *Every* Γ*-band is a* Γ*-semilattice of semigroups, which semigroups are rectangular bands.*

Proof. By Corollary 4.2 and Theorem 3.3, it is obvious. □

5. Examples

Example 5.1. Let

$$S = \left\{ a = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

and

$$\Gamma = \left\{ \alpha = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix},\ \beta = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

It is a matter of checking to see that S is a Γ-semigroup such that a and b are Γ-idempotents. In other words, S is a Γ-band. It is easy to see that S has two η_S classes; these are the one-element sets $\{a\}$ and $\{b\}$. Thus S is Γ-isomorphic to the factor semigroup S/η_S. By Theorem 3.3, S/η_S is a Γ-semilattice. Thus S is a Γ-semilattice. It can be directly shown that S is a Γ-semilattice.

Example 5.2. Let

$$S = \left\{ a = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix},\ c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

and

$$\Gamma = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

It is a matter of checking to see that S is a Γ-band, in which the classes of η_S are $\{a, b\}$ and $\{c\}$. It is easy to see that $\{a, b\}$ is a right zero Γ-semigroup.

Example 5.3. Let

$$S = \left\{ a = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix},\ c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

and

$$\Gamma = \left\{ \alpha = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix},\ \beta = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

It is a matter of checking to see that S is a Γ-band, in which the classes of η_S are $\{a, b\}$ and $\{c\}$. Moreover, $\{a, b\}$ is a right zero Γ-semigroup.

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