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Quasi-antiassociative bialgebras

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Abstract. This work explores key characteristics and properties of quasi-antiassociative algebras, including their bimodules, matched pairs and bialgebraic structures.

1. Introduction

Mock-Lie algebras, recently introduced by P. Zusmanovich in [15], are commutative algebras satisfying the Jacobi identity. These algebras have appeared in the literature under various names (see [15] and [4] for details). Two notable features of Mock-Lie algebras are: first, that algebras over the operad Koszul dual to the Mock-Lie operad can be characterized in three equivalent ways (see [15], [4]), and second, as noted in [13], Mock-Lie algebras can be derived from antiassociative algebras similarly to how they arise from associative algebras, indicating a strong link between antiassociative and Mock-Lie algebras. Here, "antiassociative algebras" refer to those where the operation $(a, b) \to ab$ satisfies $(ab)c+a(bc) = 0$ for each a, b, c. This class of algebras first appeared notably in [13], with main properties given, and in [10] where M. Markl and E. Remm (2014) established results on Koszul operads for n-ary algebras, with a focus on antiassociative operations.

In 2007, C. Bai examined left-symmetric bialgebras, which serve as analogues to Lie bialgebras, in [1]. Left-symmetric algebras (also called pre-Lie, quasi-associative, and Vinberg algebras) are Lie-admissible algebras where left multiplication operators form a Lie algebra; these were first introduced by Cayley in 1896. Bai highlighted that left-symmetric algebras also emerged in various studies in geometry and algebra, such as convex homogeneous cones [14, 9], affine manifolds, affine structures on Lie groups [11], and associative algebra deformation [6]. Notably, a Lie algebra $\mathcal{G}$ with

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compatible left-symmetric structure is the Lie algebra of a Lie group $\mathcal{G}$ with a left-invariant affine structure, where there is a left-invariant, flat, and torsion-free connection $\bigtriangledown$ in G. The left-symmetric algebra structure corresponds to the connection $\bigtriangledown$ given by $XY = \bigtriangledown_X Y$ for $X, Y \in G$, where left-symmetry reflects the connection's flatness [8, 12].

Chapoton and Livernet noted in [5] that left-symmetric algebras feature in many fields, and Burde reviewed areas where left-symmetric algebras play key roles, including vector fields, rooted tree algebras, vertex algebras, operad theory, and affine manifolds (see [3]).

This paper begins by examining a structure analogous to Lie algebras, namely the Mock-Lie structure derived from antiassociative algebras, aiming to define Mock-Lie admissible algebras as associative counterparts of Lie admissible algebras. Next, we study the structure analogous to left-symmetric and refer to it as the quasi-antiassociative structure, defined via the antiassociator of the associated bilinear product.

The goal is to define Mock-Lie admissible algebras and investigate quasi-antiassociative algebras, with particular emphasis on their bimodules, matched pairs and bialgebras.

Since quasi-antiassociative structures differ from left-symmetric structures only by a sign (see Definition 2.5), we conjecture that the quasi-antiassociative structure may also correspond to a geometric connection.

The paper is organized as follows. Section 2 reviews basic concepts and properties of antiassociative and Mock-Lie algebras, and defines Mock-Lie admissible and quasi-antiassociative algebras. Section 3 covers bimodules and matched pairs of quasi-antiassociative algebras. Section 4 studies the quasi-antiassociative bialgebras. Section 5 concludes with final remarks.

2. Definitions and main properties

Throughout this work, we consider $\mathcal{A}$ a finite dimensional vector space over the field $\mathbb{K}$ of characteristic 0 together with a bilinear product " $\cdot$ " defined as : $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $(x, y) \mapsto x \cdot y$.

Definition 2.1. [13] Let "$\cdot$" be a bilinear product in a vector space $\mathcal{A}$. Suppose that it satisfies the following law:

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z).$$

Then, we call the pair $(\mathcal{A}, \cdot)$ an *antiassociative algebra.*

Definition 2.2. [15] An algebra $(\mathcal{A}, \diamond)$ over K is called *mock-Lie* if it is commutative $x \diamond y = y \diamond x$ and satisfies the Jacobi identity:

$$(x \diamond y) \diamond z + (z \diamond x) \diamond y + (y \diamond z) \diamond x = 0$$

for any $x,y,z \in \mathcal{A}$.

Theorem 2.3. [15] *Given an antiassociative algebra $(\mathcal{A}, \cdot)$, the new algebra $\mathcal{A}^\dagger$ with multiplication give by the "anticommutator"*

$$a \diamond b = (a \cdot b + b \cdot a)\,,$$

is a mock-Lie algebra.

Definition 2.4. A *Mock-Lie admissible algebra* is a nonassociative algebra which anticommutator is a Mock-Lie algebra.

Any antiassociative algebra is a Mock-Lie admissible algebra.

Definition 2.5. $(\mathcal{A}, \cdot)$, or simply $\mathcal{A}$, is said to be a *left quasi-antiassociative algebra* if $\forall\, x, y, z\, \in \mathcal{A}$, the antiassociator of the bilinear product $\cdot$ defined by $(x, y, z)_{-1} := (x \cdot y) \cdot z + x \cdot (y \cdot z)$, is symmetric in x and y, i.e.

$$(x, y, z)_{-1} = -(y, x, z)_{-1}.$$

As matter of notation simplification, we will denote $x \cdot y$ by xy if not any confusion.

Proposition 2.6. *Any antiassociative algebra is a left quasi-antiassociative algebra, and*

$$\mu \circ (\mu \otimes id) + \mu \circ (id \otimes \mu) + \mu \circ ((\mu \circ \tau) \otimes id) + \mu(id \otimes (\mu \circ \tau)) = 0,$$

where μ is the multiplication operator ant τ is such that $\tau(x \otimes y) = y \otimes x$.

Proof. Let $(\mathcal{A}, \cdot)$ be an antiassociative algebra. We have $\forall x, y, z \in \mathcal{A}$

$$(x \cdot y) \cdot z = -x \cdot (y \cdot z) \Leftrightarrow (x, y, z)_{-1} = 0 = -(y, x, z)_{-1}.$$

Thus $(x, y, z)_{-1} + (y, x, z)_{-1} = 0$, which implies that

$$(xy)z + x(yz) + (yx)z + y(xz) = 0.$$

Supposing μ be the bilinear product on $\mathcal{A}$, then the previous relation can be written as

$$\mu(\mu(x \otimes y) \otimes z) + \mu(x \otimes \mu(y \otimes z)) + \mu((\mu \circ \tau)(x \otimes y) \otimes z) + \mu(y \otimes (\mu \circ \tau) z \otimes x) = 0$$
$$\Leftrightarrow \mu \circ (\mu \otimes id) + \mu \circ (id \otimes \mu) + \mu \circ ((\mu \circ \tau) \otimes id) + \mu(id \otimes (\mu \circ \tau)) = 0. \quad \square$$

Definition 2.7. An algebra $(\mathcal{A}, \cdot)$ over $\mathbb{K}$ with the bilinear product given by $(x, y) \mapsto x \cdot y$ is called *right quasi-antiassociative* if the associator associated to the bilinear product on $\mathcal{A}$ is symmetric in right, i.e. for all $x, y, z \in \mathcal{A}$

$$(x, y, z)_{-1} = -(x, z, y)_{-1}.$$

Theorem 2.8. *The opposite algebra of the left quasi-antiassociative algebra is a right quasi-antiassociative algebra. The opposite algbera of the right quasi-antiassociative algebra is a left quasi-antiassociative algebra.*

Proof. Suppose that $(\mathcal{A}, \cdot)$ is a left quasi-antiassociative algebra and $(\mathcal{A}, \circ)$ the opposite algebra of the algebra $\mathcal{A}$. For any $x, y, z \in \mathcal{A}$ we have:

$$\begin{aligned}(x, y, z)_{-1,\circ} &= (x \circ y) \circ z + x \circ (y \circ z) = z \cdot (y \cdot x) + (z \cdot y) \cdot x \\ &= (z, y, x)_{-1} = -(y, z, x)_{-1} = -y \cdot (z \cdot x) - (y \cdot z) \cdot x \\ &= -(x \circ z) \circ y - x \circ (z \circ y) = -(x, z, y)_{-1,\circ}.\end{aligned}$$

Therefore, for all $x, y, z \in \mathcal{A}$, $(x, y, z)_{-1,\circ} = -(x, z, y)_{-1,\circ}$. Conversely, if $(\mathcal{A}, \circ)$ is a right quasi-antiassociative algebra, then by the same way we easily show that the algebra $(\mathcal{A}, \cdot)$ is left quasi-antiassociative. $\square$

Theorem 2.8 proves the equivalence between left quasi-antiassociative algebras and right quasi-antiassociative algebras. Thus, for the rest of this paper, without any further clarification, left quasi-antiassociative algebras are called quasi-antiassociative algebras.

Considering the representations of the left L and right R multiplication operations:

$$\begin{array}{rcl} L : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}) \\ x & \longmapsto & L_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & x \cdot y, \end{array} \end{array}$$

$$\begin{array}{rcl} R : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}) \\ x & \longmapsto & R_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & y \cdot x, \end{array} \end{array}$$

we infer the adjoint representation $\mathrm{ad} := L+R$ of the sub-adjacent Mock-Lie algebra of a quasi-antiassociative algebra $\mathcal{A}$ as follows:

$$\begin{array}{rcl} \mathrm{ad} : \mathcal{A} & \longrightarrow & \mathfrak{gl}(\mathcal{A}) \\ x & \longmapsto & \mathrm{ad}_x : \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ y & \longmapsto & [x, y], \end{array} \end{array}$$

such that $\forall\, x, y \in \mathcal{A}, \mathrm{ad}_x(y) := (L_x + R_x)(y)$.

Definition 2.9. [15] A vector space V is a *module over a Mock-Lie algebra* $\mathcal{A}$, if there is a linear map (a representation) $\rho : \mathcal{A} \to End(V)$ such that

$$\rho(x \diamond y)(v) = -\rho(x)(\rho(y)v) - \rho(y)(\rho(x)v)$$

for any $x, y \in \mathcal{A}$ and $v \in V$.

Proposition 2.10. *Let $(\mathcal{A}, \cdot)$ be a quasi-antiassociative algebra. For any $x, y \in \mathcal{A}$, the following relations are satisfied:*

- *The anticommutator associated to the bilinear product " $\cdot$ " given by $[x, y] = x \cdot y + y \cdot x$ defines a Mock-Lie algebra structure on $\mathcal{A}$.*
- *The left multiplication operator gives a representation of the Mock-Lie algebra, that is*
$$L_{[x,y]} = -[L_x, L_y],$$
and the following relation is also satisfied
$$[L_x, R_y] = -R_{x \cdot y} - R_y R_x,$$
where the linear map R is the right multiplication operator associated to the bilinear product " $\cdot$ " on $\mathcal{A}$.
- $[L_x, R_y] = -[R_x, L_y]$
- $L_{xy} + L_x L_y = -L_{yx} - L_y L_x,$
- *$ad = L + R$ a linear representation of the sub-adjacent Mock-Lie algebra of $(\mathcal{A}, \cdot)$ and,*
$$[ad_x, ad_y] = ad_{[x,y]}.$$

Proof. Consider the quasi-antiassociative algebra $(\mathcal{A}, \cdot)$. For any $x, y, z \in \mathcal{A}$ we have

$$\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [y, x]] = [x, yz + zy] + [y, zx + xz] + [z, xy + yx] \\
&= x(yz) + x(zy) + (yz)x + (zy)x + y(zx) + y(xz) + (zx)y + (xz)y + z(xy) \\
&\qquad + z(yx) + (xy)z + (yx)z \\
&= \{x(yz) + (yz)x + y(zx) + (zx)y + z(xy) + (xy)z\} \\
&\qquad + \{(zy)x + x(zy) + y(xz) + (xz)y + z(yx) + (yx)z\} \\
&= \{(x, y, z)_{-1} + (y, z, x)_{-1} + (z, x, y)_{-1}\} \\
&\qquad + \{(z, y, x)_{-1} + (x, z, y)_{-1} + (y, x, z)_{-1}\} \\
&= \{(y, x, z)_{-1} + (x, y, z)_{-1}\} + \{(z, y, x)_{-1} + (y, z, x)_{-1}\} + \{(x, z, y)_{-1}
\end{aligned}$$

$$+(z,x,y)_{-1}\} = 0.$$

On other hand, we have for all $x, y, z \in \mathcal{A}$

$$\begin{aligned}(x,y,z)_{-1} = -(y,x,z)_{-1} &\Leftrightarrow (xy)z + x(yz) = -(yx)z - y(xz)\\ &\Leftrightarrow (xy)z + (yx)z = -x(yz) - y(xz)\\ &\Leftrightarrow (L_{xy} + L_{yx})(z) = -(L_xL_y + L_yL_x)(z)\\ &\Leftrightarrow (L_{xy} + L_{yx})(z) = -[L_x, L_y](z)\end{aligned}$$

Therefore, $L_{[x,y]} = -[L_x, L_y]$ for all $x, y \in \mathcal{A}$ holds.

We have for all $x, y, z \in \mathcal{A}$

$$\begin{aligned}(x,y,z)_{-1} = -(y,x,z)_{-1} &\Leftrightarrow (xy)z + x(yz) = -(yx)z - y(xz)\\ &\Leftrightarrow (R_zL_x - L_xR_z)(y) = -(R_zR_x + R_{xz})(y)\\ &\Leftrightarrow [R_z, L_x](y) = -(R_zR_x + R_{xz})(y),\end{aligned}$$

Therefore, the following relation holds

$$[L_x, R_y] = -(R_{xy} + R_yR_x).$$

We have for all $x, y, z \in \mathcal{A}$

$$\begin{aligned}[L_x, R_y](z) &= L_x\left(R_y(z)\right) + R_y\left(L_x(z)\right)\\ &= x(zy) + (xz)y = (x,z,y)_{-1} = -(z,x,y)_{-1}\\ &= -\left((zx)y + z(xy)\right) = -\left(R_y\left(R_x(z)\right) + R_{xy}(z)\right)\\ &= -\left(R_yR_x + R_{xy}\right)(z)\\ &= (z,y,x)_{-1} = (zy)x + z(yx) = R_xR_y(z) + R_{xy}(z)\\ &= -(y,z,x) - (yz)x - y(zx) = -R_xL_y(z) - L_yR_x(z)\\ &= -[R_x, L_y](z).\end{aligned}$$

Therefore $[L_x, R_y] = -[R_x, L_y]$.

We have for all $x, y, z \in \mathcal{A}$

$$\begin{aligned}\left(L_{xy} + L_xL_y\right)(z) &= L_{xy}(z) + L_x\left(L_y(z)\right) = (xy)z + x(yz)\\ &= (x,y,z)_{-1} = -(y,x,z)_{-1} = -\left((yx)z + y(xz)\right)\\ &= -\left(L_{yx}(z) + L_y(L_x(z))\right)\\ &= -\left(L_{yx} + L_yL_x\right)(z).\end{aligned}$$

Thus $L_{xy} + L_xL_y = -L_{yx} - L_yL_x$.

We have for all $x, y \in \mathcal{A}$,

$$\begin{aligned}
[ad_x, ad_y] &= [L_x, L_y] + [L_x, R_y] + [R_x, L_y] + [R_x, R_y] \\
&= [L_x, L_y] + [R_x, R_y] \\
&= (L_x L_y + R_x R_y) + (L_y L_x + R_y R_x) \\
&= (L+R)_{xy} + (L+R)_{yx} = (L+R)_{xy+yx} \\
&= (L+R)_{[x,y]} = ad_{[x,y]}
\end{aligned}$$

This completes the proof. □

Remark 2.11. From this proposition, quasi-antiassociative algebras are Mock-Lie admissible algebras.

Thus, $(\mathcal{A}, \cdot)$ can be called the compatible quasi-antiassociative algebra product of the Mock-Lie algebra $\mathcal{G}(\mathcal{A})$.

3. Bimodules and matched pairs

Definition 3.1. Let $\mathcal{A}$ be a quasi-antiassociative algebra, V be a vector space. Suppose $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps such that

$$[l_x, r_y] = -[l_y, r_x] \tag{3.1}$$

$$l_{xy} + l_x l_y = -l_{yx} - l_y l_x \tag{3.2}$$

for all $x, y \in \mathcal{A}$. Then, (l, r, V) (or simply (l, r)) is called *bimodule of the quasi-antiassociative algebra $\mathcal{A}$.*

Proposition 3.2. *Let $(\mathcal{A}, \cdot)$ be a quasi-antiassociative algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$. Then, (l, r, V) is a bimodule of $\mathcal{A}$ if and only if, the semidirect sum $\mathcal{A} \oplus V$ of vector spaces is turned into a quasi-antiassociative algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $\forall x_1, x_2 \in \mathcal{A},\ v_1, v_2 \in V$,*

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1).$$

We denote it by $\mathcal{A} \ltimes^{-1}_{l,r} V$ or simply $\mathcal{A} \ltimes^{-1} V$.

Proof. It is obvious that the semidirect sum of two vector spaces is also a vector space. Now suppose that (l, r, V) is a bimodule of $\mathcal{A}$ and show that $(\mathcal{A} \oplus V, *)$ is a quasi-antiassociative algebra. Since $*$ is a bilinear product, for all $x_1, x_2, x_3 \in \mathcal{A}$ and for all $v_1, v_2, v_3 \in V$, we have:

$$\begin{aligned}
&((x_1+v_1),(x_2+v_2),(x_3+v_3))_{-1}\\
&\quad=\{(x_1+v_1)*(x_2+v_2)\}*(x_3+v_3)+(x_1+v_1)*\{(x_2+v_2)*(x_3+v_3)\}\\
&\quad=(x_1x_2)x_3+l_{x_1x_2}v_3+r_{x_3}(l_{x_1}v_2+r_{x_2}v_1)+x_1(x_2x_3)+l_{x_1}(l_{x_2}v_3)\\
&\qquad\qquad\qquad\qquad+l_{x_1}(r_{x_3}v_2)+r_{x_2x_3}v_1\\
&\quad=(x_1x_2)x_3+l_{x_1x_2}v_3+r_{x_3}(l_{x_3}v_2)+r_{x_3}(r_{x_2}v_1)+x_1(x_2x_3)+l_{x_1}(l_{x_2}v_3)\\
&\qquad\qquad\qquad\qquad+l_{x_1}(r_{x_3}v_2)+r_{x_2x_3}v_1,
\end{aligned}$$

$$\begin{aligned}
&((x_1+v_1),(x_2+v_2),(x_3+v_3))\\
&\quad=(x_1,x_2,x_3)_{-1}+(l_{x_1x_2}+l_{x_1}l_{x_2})v_3+(r_{x_3}l_{x_1}+l_{x_1}r_{x_3})v_2+(r_{x_3}r_{x_2}+r_{x_2x_3})v_1.\\
&((x_2+v_2),(x_1+v_1),(x_3+v_3))\\
&\quad=(x_2,x_1,x_3)+(l_{x_2x_1}+l_{x_2}l_{x_1})v_3+(r_{x_3}l_{x_2}+l_{x_2}r_{x_3})v_1+(r_{x_3}r_{x_1}+r_{x_1x_3})v_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(x_1+v_1,x_2+v_2,x_3+v_3)_{-1}&=-(x_3+v_3,x_2+v_2,x_1+v_1)_{-1}\\
&\Leftrightarrow\begin{cases} l_{x_1x_2}+l_{x_1}l_{x_2}=-l_{x_2x_1}-l_{x_2}l_{x_1}\\ r_{x_3}l_{x_1}+l_{x_1}r_{x_3}=-r_{x_3}r_{x_1}-r_{x_1x_3}\\ r_{x_2x_3}+r_{x_3}r_{x_2}=-r_{x_3}l_{x_2}-l_{x_2}r_{x_3}\end{cases}\\
&\Leftrightarrow(\mathcal{A}\oplus V,*)\text{ is a quasi-antiassociative algebra.}\qquad\square
\end{aligned}$$

Furthermore, we derive the next result.

Proposition 3.3. *Let $\mathcal{A}$ be a quasi-antiassociative algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l,r:\mathcal{A}\to\mathfrak{gl}(V)$, such that (l,r,V) is a bimodule of $\mathcal{A}$. Then, the map: $l+r:\mathcal{A}\longrightarrow\mathfrak{gl}(V)$ $x\longmapsto l_x+r_x$, is a linear representation of the sub-adjacent Mock-Lie algebra of $\mathcal{A}$.*

Proof. Let (l,r,V) be a bimodule of the quasi-antiassociative algebra $\mathcal{A}$. Then, $\forall x,y\in\mathcal{A}\quad[l_x,r_y]=-[l_y,r_x]; l_{xy}+l_xl_y=-l_yl_x-l_{yx}$. Besides, it is a matter of straightforward computation to show that $l+r$ is a linear map on $\mathcal{A}$. Then, we have:

$$\begin{aligned}
[(l+r)(x),(l+r)(y)]&=[l_x+r_x,l_y+r_y]=[l_x,l_y]+[l_x,r_y]+[r_x,l_y]+[r_x,r_y]\\
&=[l_x,l_y]+[r_x,r_y]\\
&=l_xl_y+l_yl_x+r_xr_y+r_yr_x\\
&=\{l_xl_y+r_xr_y\}+\{l_yl_x+r_yr_x\}\\
&=\{l_{xy}+r_{xy}\}+\{l_{yx}+r_{yx}\}=(l+r)_{xy}+(l+r)_{yx}\\
&=(l+r)_{[x,y]}.
\end{aligned}$$

Therefore, (l,r,V) is a bimodule of $\mathcal{A}$ implies that $l+r$ is a representation of the linear representation of the sub-adjacent Mock-Lie algebra of $\mathcal{A}$. $\square$

Example 3.4. *According to the Proposition 2.10, one can deduce that $(L, R, \mathcal{A})$ is a bimodule of the quasi-antiassociative algebra $\mathcal{A}$, where L and R are the left and right multiplication operator representations, respectively.*

Definition 3.5. Let $\mathcal{G}$ and $\mathcal{H}$ be two Mock-Lie algebras and let $\mu : \mathcal{H} \to \mathfrak{gl}(\mathcal{G})$ and $\rho : \mathcal{G} \to \mathfrak{gl}(\mathcal{H})$ be two Mock-Lie algebra representations. Then, $(\mathcal{G}, \mathcal{H}, \rho, \mu)$ is called a *matched pair* of the Mock-Lie algebras $\mathcal{G}$ and $\mathcal{H}$, denoted by $\mathcal{H} \bowtie_{\mu}^{-1,\rho} \mathcal{G}$ if and only if μ and ρ satisfy: for all $x, y \in \mathcal{G}, a, b \in \mathcal{H}$,

$$\rho(x)\,[a, b] + [\rho(x)a, b] + [a, \rho(x)b] + \rho(\mu(a)x)b + \rho(\mu(b)x)a = 0, \tag{3.3}$$

$$\mu(a)\,[x, y] + [\mu(a)x, y] + [x, \mu(a)y] + \mu(\rho(x)a)y + \mu(\rho(y)a)x = 0. \tag{3.4}$$

In this case, $(\mathcal{G} \oplus \mathcal{H}, [,])$ defines a Mock-Lie algebra with respect to the product $*$ satisfying:

$$[(x + a), (y + b)] = [x, y] + \mu(a)y + \mu(b)x + [a, b] + \rho(x)b + \rho(y)a.$$

Theorem 3.6. *Let $(\mathcal{A}, \cdot)$ and $(\mathcal{B}, \circ)$ be two quasi-antiassociative algebras. Suppose that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodules of $\mathcal{A}$ and $\mathcal{B}$, respectively, obeying the relations:*

$$r_{\mathcal{A}}(x)([a, b]) = r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + a \circ (r_{\mathcal{A}}(x)b) + b \circ (r_{\mathcal{A}}(x)a), \tag{3.5}$$

$$\begin{aligned} -l_{\mathcal{A}}(x)(a \circ b) \quad &= \quad l_{\mathcal{A}}(l_{\mathcal{B}}(a)x + r_{\mathcal{B}}(a)x)b + (l_{\mathcal{A}}(x)a + r_{\mathcal{A}}(x)a) \circ b \\ &+ \quad r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + a \circ (l_{\mathcal{A}}(x)b), \end{aligned} \tag{3.6}$$

$$r_{\mathcal{A}}(a)[x, y] = r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + x(r_{\mathcal{B}}(a)y) + y(r_{\mathcal{B}}(a)x), \tag{3.7}$$

$$\begin{aligned} -l_{\mathcal{B}}(a)(xy) \quad &= \quad l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + (r_{\mathcal{B}}(a)x)y + x(l_{\mathcal{B}}(a)y) + r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x \\ &+ \quad (l_{\mathcal{B}}(a)x)y + l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y, \end{aligned} \tag{3.8}$$

for all $x, y \in A$ and $a, b \in \mathcal{B}$. Then, there is a quasi-antiassociative algebra structure on $\mathcal{A} \oplus \mathcal{B}$ given by:

$$(x+a) * (y+b) = (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a). \tag{3.9}$$

We denote this quasi-antiassociative algebra by $\mathcal{A} \bowtie_{l_{\mathcal{B}}, r_{\mathcal{B}}}^{-1, l_{\mathcal{A}}, r_{\mathcal{A}}} \mathcal{B}$, or simply by $\mathcal{A} \bowtie^{-1} \mathcal{B}$. Then $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ satisfying the above conditions is called matched pair of the quasi-antiassociative algebras $\mathcal{A}$ and $\mathcal{B}$.

Proof. Consider $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$. We have

$$\begin{aligned} (x + a) * (y + b) &= (xy + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a), \\ x * a &= r_{\mathcal{B}}(a)x + l_{\mathcal{A}}(x)a, \;\; b = 0, y = 0, \\ a * y &= l_{\mathcal{B}}(a)x + r_{\mathcal{A}}(y)a, \;\; x = 0, b = 0. \end{aligned}$$

Since $(x,y,z)_{-1} = (xy)z + x(yz)$, we have

$$\begin{aligned}(x,b,c)_{-1} &= (x*b)*c + x*(b\circ c)(r_{\mathcal{B}}(b)x + l_{\mathcal{A}}(x)x)*c + x*(b\circ c)\\ &= r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x) + (l_{\mathcal{A}}(x)b)\circ c + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c + r_{\mathcal{B}}(b\circ c)x + l_{\mathcal{A}}(x)(b\circ c),\end{aligned}$$

$$\begin{aligned}(a,y,c)_{-1} =& \ (a*y)*c + a*(y*c)\\ =& \ (l_{\mathcal{B}}(a)y + r_{\mathcal{A}}(y)a)*c + a*(l_{\mathcal{A}}(y)c + r_{\mathcal{B}}(c)y)\\ =& \ l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c + r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y) + (r_{\mathcal{A}}(y)a)\circ c\\ +& \ a\circ(l_{\mathcal{A}}(y)c) + l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y) + r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a,\end{aligned}$$

$$\begin{aligned}(x,y,c)_{-1} &= (x\cdot y)*c + x*(y*c)\\ &= (l_{\mathcal{A}}(x\cdot y)c + r_{\mathcal{B}}(c)(x\cdot y)*c + x*(l_{\mathcal{A}}(y)c + r_{\mathcal{B}}(c)y)\\ &= l_{\mathcal{A}}(xy)c + r_{\mathcal{B}}(c)(xy) + x\cdot(r_{\mathcal{B}}(c)y) + l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c) + r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x,\end{aligned}$$

$$\begin{aligned}(x,b,z)_{-1} =& \ (x*b)*z + x*(b*z)\\ =& \ (l_{\mathcal{A}}(x)b + r_{\mathcal{B}}(b)x)*z + x*(l_{\mathcal{B}}(b)z + r_{\mathcal{A}}(z)b)\\ =& \ l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z + r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b) + (r_{\mathcal{B}}(b)x)\cdot z\\ +& \ x\cdot(l_{\mathcal{B}}(b)z) + l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b) + r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x,\end{aligned}$$

$$(a,b,c)_{-1} = (a\circ b)\circ c + a\circ(b\circ c)$$

$$\begin{aligned}(a,y,z)_{-1} &= (a*y)*z + a*(y\cdot z)\\ &= (l_{\mathcal{B}}(a)y + r_{\mathcal{A}}(y)a)*z + l_{\mathcal{B}}(a)(y\cdot z) + r_{\mathcal{A}}(y\cdot z)a\\ &= (l_{\mathcal{B}}(a)y)z + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z + r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a) + l_{\mathcal{B}}(a)(y\cdot z) + r_{\mathcal{A}}(y\cdot z)a,\end{aligned}$$

$$\begin{aligned}(a,b,z)_{-1} &= (a\circ b)*z + a*(b*z)\\ &= (l_{\mathcal{B}}(a\circ a)\cdot z + r_{\mathcal{A}}(z)(a\circ b)) + a*(l_{\mathcal{B}}(b)z + r_{\mathcal{A}}(z)b)\\ &= l_{\mathcal{B}}(a\circ b)z + r_{\mathcal{A}}(z)(\circ b) + a\circ(r_{\mathcal{A}}(z)b) + l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z) - r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a,\end{aligned}$$

The first part of the associator reads:

$$\begin{aligned}&\{(x+a)*(y+b)\}*(z+c)\\ &= \{(xy + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a\circ b + l_{\mathcal{A}}(y) + r_{\mathcal{A}}(y)a)\}*(z+c)\\ &= (xy + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x)z + l_{\mathcal{B}}(a\circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)z\\ &\quad + r_{\mathcal{B}}(c)(xy + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a\circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)\circ c\\ &\quad + l_{\mathcal{A}}(a)(yz + l_{\mathcal{B}}(b)z + r_{\mathcal{B}}(c)y) + r_{\mathcal{A}}(z)(a\circ b l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)\\ &= (xy)z + (l_{\mathcal{B}}(a)y)z + (r_{\mathcal{B}}(b)x)z + l_{\mathcal{B}}(a\circ b)z + l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z\\ &\quad + r_{\mathcal{B}}(c)(xy) + r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y) + r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)) + (a\circ b)\circ c + (l_{\mathcal{A}}(x)b)\circ c\\ &\quad + (r_{\mathcal{A}}(y)a)\circ c + l_{\mathcal{A}}(a)(yz) + l_{\mathcal{A}}(a)(l_{\mathcal{B}}(b)z) + l_{\mathcal{A}}(a)(r_{\mathcal{B}}(c)y)\\ &\quad + r_{\mathcal{A}}(z)(a\circ b) + r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b) + r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)\end{aligned}$$

while its second part:

$$(x+a)*\{(y+b)*(z+c)\}$$

$$
\begin{aligned}
&= (x+a)*\{yz+l_{\mathcal{B}}(b)z+r_{\mathcal{B}}(c)y+b\circ c+l_{\mathcal{A}}(y)c+r_{\mathcal{A}}(z)b\}\\
&= x(yz+l_{\mathcal{B}}(b)z+r_{\mathcal{B}}(c)y)+l_{\mathcal{B}}(a)(yz+l_{\mathcal{B}}(b)z+r_{\mathcal{B}}(c)y)\\
&\quad +r_{\mathcal{B}}(b\circ c+l_{\mathcal{A}}(y)c+r_{\mathcal{A}}(z)b)x+a\circ(b\circ c+l_{\mathcal{A}}(y)c\\
&\quad +r_{\mathcal{A}}(z)b)+l_{\mathcal{A}}(x)(b\circ c+l_{\mathcal{A}}(y)c+r_{\mathcal{A}}(z)b)+r_{\mathcal{A}}(yz+l_{\mathcal{B}}(b)z+r_{\mathcal{B}}(c)y)a\\
&= x(yz)+x(l_{\mathcal{B}}(b)z)+x(r_{\mathcal{B}}(c)y)+l_{\mathcal{B}}(a)(yz)+l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)\\
&\quad +l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)+r_{\mathcal{B}}(b\circ c)x+r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x+r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x\\
&\quad +a\circ(b\circ c)+a\circ(l_{\mathcal{A}}(y)c)+a\circ(r_{\mathcal{A}}(z)b)+l_{\mathcal{A}}(x)(b\circ c)\\
&\quad +l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c)+l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b)+r_{\mathcal{A}}(yz)a+r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a+r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a
\end{aligned}
$$

and the associator takes the form:

$$
\begin{aligned}
&(x+a,y+b,z+c)_{-1}\\
&= \{(xy)z-x(yz)\}+\{(a\circ b)\circ c-a\circ(b\circ c)\}\\
&\quad +\{r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)+(l_{\mathcal{A}}(x)b)\circ c+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c\\
&\quad -r_{\mathcal{B}}(b\circ c)x-l_{\mathcal{A}}(x)(b\circ c)\}\{l_{\mathcal{A}}(l_{\mathcal{B}}()a)yc+r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)+(r_{\mathcal{A}}y)\circ c\\
&\quad -a\circ(l_{\mathcal{A}}(y)c)-l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)-r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a\}+\{l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c\\
&\quad +r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)+(r_{\mathcal{A}}(y)a)\circ c-a\circ(l_{\mathcal{A}}(y)c)-l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)\\
&\quad -r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a\}+\{l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z+r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b)+(r_{\mathcal{B}}(b)x)z\\
&\quad -x(l_{\mathcal{B}}(b)z)-l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b)x\}+\{(l_{\mathcal{B}}(a)y)z+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z\\
&\quad +r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)-l_{\mathcal{B}}(a)(yz)-r_{\mathcal{A}}(yz)a\}+\{l_{\mathcal{B}}(a\circ b)z\\
&\quad +r_{\mathcal{A}}(z)(a\circ b)-a\circ(r_{\mathcal{A}}(z)b)-l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)-r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\}\\
&= (x,y,z)_{-1}+(a,b,c)_{-1}+\{r_{\mathcal{B}}(c)(x\cdot y)+l_{\mathcal{A}}(x\cdot y)c+x\cdot(r_{\mathcal{B}}(c)y)\\
&\quad +l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c)+r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x\}+\{r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c\\
&\quad +r_{\mathcal{B}}(b\circ c)x+(l_{\mathcal{A}}(x)b)\circ c+l_{\mathcal{A}}(x)(b\circ c)\}+\{(r_{\mathcal{B}}(b)x)\cdot z+l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z\\
&\quad +r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b)+x\cdot(l_{\mathcal{B}}(b)z)+r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x+l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b)\}\\
&\quad +\{(l_{\mathcal{B}}(a)y)\cdot z+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z+r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)+l_{\mathcal{B}}(a)(y\cdot z)\\
&\quad +r_{\mathcal{A}}(y\cdot z)a\}+\{r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)+(r_{\mathcal{A}}(y)a)\circ c+l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c\\
&\quad +l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)+a\circ(l_{\mathcal{A}}(y)c)+r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a\}+\{l_{\mathcal{B}}(a\circ b)z\\
&\quad +r_{\mathcal{A}}(z)(a\circ b)+l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)+a\circ(r_{\mathcal{A}}(z)b)+r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\}.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
(x,y,c)_{-1} &= r_{\mathcal{B}}(c)(x\cdot y)+x\cdot(r_{\mathcal{B}}(c)y)+r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x+l_{\mathcal{A}}(x\cdot y)c+l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c),\\
(x,b,c)_{-1} &= r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c+(l_{\mathcal{A}}(x)b)\circ c+r_{\mathcal{B}}(b\circ c)x+l_{\mathcal{A}}(x)(b\circ c),\\
(x,b,z)_{-1} &= (r_{\mathcal{B}}(b)x)\cdot z+l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z+r_{\mathcal{A}}(z)(l_{\mathcal{A}}(x)b)+x\cdot(l_{\mathcal{B}}(b)z)\\
&\quad +r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x+l_{\mathcal{A}}(x)(r_{\mathcal{A}}(z)b),\\
(a,y,z)_{-1} &= (l_{\mathcal{B}}(a)y)\cdot z+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z+r_{\mathcal{A}}(z)(r_{\mathcal{A}}(y)a)+l_{\mathcal{B}}(a)(y\cdot z)+r_{\mathcal{A}}(y\cdot z)a,\\
(a,y,c)_{-1} &= r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)+(r_{\mathcal{A}}(y)a)\circ c+l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c+l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)\\
&\quad +a\circ(l_{\mathcal{A}}(y)c)+r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a,\\
(a,b,z)_{-1} &= l_{\mathcal{B}}(a\circ b)z+r_{\mathcal{A}}(z)(a\circ b)+l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)+r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a+r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\\
&\quad +a\circ(r_{\mathcal{A}}(z)b),
\end{aligned}
$$

which can also be re-expressed as:

$$\begin{aligned}(x+a, y+b, z+c)_{-1} = {} & (x,y,z)_{-1} + (x,y,c)_{-1} + (x,b,z)_{-1} + (x,b,c)_{-1} \\ & +(a,y,z)_{-1} + (a,y,c)_{-1} + (a,b,z)_{-1} + (a,b,c)_{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}(y+b, x+a, z+c)_{-1} = {} & (y,x,z)_{-1} + (b,x,z)_{-1} + (y,x,c_{-1}) + (b,a,z)_{-1} \\ & +(y,a,z)_{-1} + (y,a,c)_{-1} + (b,a,c)_{-1} + (b,x,c)_{-1}.\end{aligned}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are quasi-antiassociative algebras, we have

$$\begin{aligned}(x,y,z)_{-1} &= -(y,x,z)_{-1} \\ (a,b,c)_{-1} &= -(b,a,c)_{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}&(x,b,z)_{-1} = -(b,x,z)_{-1} \Leftrightarrow (y,a,z)_{-1} = -(a,y,z)_{-1}\{x \to y, b \to a, z \to z\} \\ &(x,b,c)_{-1} = -(b,x,c)_{-1} \Leftrightarrow (a,y,c)_{-1} = -(y,a,c)_{-1}\{x \to y, b \to a, c \to c\}.\end{aligned}$$

Then, it remains to show that:

$$(x,a,y)_{-1} = -(a,x,y)_{-1}, \tag{3.10}$$

$$(x,a,b)_{-1} = -(a,x,b)_{-1}, \tag{3.11}$$

$$(x,y,a)_{-1} = -(y,x,a)_{-1}, \tag{3.12}$$

$$(a,b,x)_{-1} = -(b,a,x)_{-1}. \tag{3.13}$$

We have

$$\begin{aligned}(3.10) \Leftrightarrow {} & l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + r_{\mathcal{A}}(y)(l_{\mathcal{A}}(x)a) + (r_{\mathcal{B}}(a)x)y + x(l_{\mathcal{B}}(a)y) \\ & +l_{\mathcal{A}}(x)(r_{\mathcal{A}}(y)a) + r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x \\ = {} & -\{(l_{\mathcal{B}}(a)x)y + l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y + r_{\mathcal{A}}(y)(r_{\mathcal{A}}(x)a) \\ & +l_{\mathcal{B}}(a)(xy) + r_{\mathcal{A}}(xy)a\} \\ \Leftrightarrow {} & -l_{\mathcal{B}}(a)(xy) = l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + (r_{\mathcal{B}}(a)x)y + x(l_{\mathcal{B}}(a)y) + \\ & +r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + (l_{\mathcal{B}}(a)x)y + l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y + l_{\mathcal{B}}(a)(xy)\end{aligned}$$

$(3.10) \Leftrightarrow (3.8) \quad \text{since}[l_x, r_y] = -r_{x \cdot y} - r_y r_x.$

$$\begin{aligned}(3.11) \Leftrightarrow {} & r_{\mathcal{B}}(b)(r_{\mathcal{B}}(a)x) + (l_{\mathcal{A}}(x)a) \circ b + l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b + r_{\mathcal{B}}(a \circ b)x \\ & +l_{\mathcal{A}}(x)(a \circ b) = -\{l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + r_{\mathcal{B}}(b)(l_{\mathcal{B}}(a)x) + (r_{\mathcal{A}}(x)a) \circ b \\ & +a \circ (l_{\mathcal{A}}(x)b) + l_{\mathcal{B}}(a)(r_{\mathcal{B}}(b)x) + r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a\} \\ \Leftrightarrow {} & -l_{\mathcal{A}}(x)(a \circ b) = l_{\mathcal{A}}(l_{\mathcal{B}}(a)x + r_{\mathcal{B}}(a)x)b + (l_{\mathcal{A}}(x)a + r_{\mathcal{A}}(x)a) \circ b \\ & +r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + a \circ (l_{\mathcal{A}}(x)b) \text{ with, } (l_{\mathcal{B}}, r_{\mathcal{B}}) \text{ is bimodule of } \mathcal{B}\end{aligned}$$

$(3.11) \Leftrightarrow (3.6)$ with, $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is bimodule of $\mathcal{B}$.

$$(3.12) \Leftrightarrow l_{\mathcal{A}}(xy)a + r_{\mathcal{B}}(a)(xy) + x(r_{\mathcal{B}}(a)y) + l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)a) + r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x$$

$$= -\{l_{\mathcal{A}}(yx)a + r_{\mathcal{B}}(a)(yx) + y(r_{\mathcal{B}}(a)x) + l_{\mathcal{A}}(y)(l_{\mathcal{A}}(x)a) + r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y\}$$
$$\Leftrightarrow r_{\mathcal{A}}(a)[x,y] = r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y + x(r_{\mathcal{B}}(a)y) + y(r_{\mathcal{B}}(a)x)$$
$$(3.12) \Leftrightarrow (3.7), \text{since } (3.2) \text{ hold.}$$

$$(3.13) \Leftrightarrow l_{\mathcal{B}}(a\circ b)x + r_{\mathcal{A}}(x)(a\circ b) + a\circ(r_{\mathcal{A}}(x)b) + l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)x) + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a$$
$$= -\{l_{\mathcal{B}}(b\circ a)x + r_{\mathcal{A}}(x)(b\circ a) + b\circ(r_{\mathcal{A}}(x)a) + l_{\mathcal{B}}(b)(l_{\mathcal{B}}(a)x) + r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b\}$$
$$(3.13) \Leftrightarrow r_{\mathcal{A}}(x)([a,b]) = r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b + a\circ(r_{\mathcal{A}}(x)b) + b\circ(r_{\mathcal{A}}(x)a) \text{ since } (3.2) \text{ hold,}$$

(3.13) $\Leftrightarrow$ (3.5) and $l_{\mathcal{B}}$ is a linear representation of the sub-adjacent Mock-Lie algebra $\mathcal{G}(\mathcal{B})$.

Hence, $\mathcal{A} \bowtie^{-1} \mathcal{B}$ is a quasi-antiassociative algebra if and only if $(l_{\mathcal{A}}, r_{\mathcal{A}})$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is a bimodule of $\mathcal{B}$ and equations (3.5) – (3.8) hold.

On the other hand, if $\mathcal{A}$ and $\mathcal{B}$ are quasi-antiassociative sub-algebras of a quasi-antiassociative algebra C such that $C = \mathcal{A} \oplus \mathcal{B}$ which is a direct sum of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$, then the linear maps

$$l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to \mathfrak{gl}(\mathcal{B}), \quad l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to \mathfrak{gl}(\mathcal{A}).$$

defined by

$$x * a = l_{\mathcal{A}}(x)a + r_{\mathcal{B}}(a)x$$
$$a * x = l_{\mathcal{B}}(a)x + r_{\mathcal{A}}(x)a$$

satisfy the equations (3.5) – (3.8). In addition, $(l_{\mathcal{A}}, r_{\mathcal{A}})$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}})$ is a bimodule of $\mathcal{B}$. □

Corollary 3.7. *Let $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ be a matched pair of quasi-antiassociative algebras. Then, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), l_{\mathcal{A}} + r_{\mathcal{A}}, l_{\mathcal{B}} + r_{\mathcal{B}})$ is a matched pair of sub-adjacent Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$.*

Proof. By using the Proposition 3.3 and the bimodules $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$, we have: $\mathrm{ad}_{\mathcal{A}} := l_{\mathcal{A}} + r_{\mathcal{A}}$ and $\mathrm{ad}_{\mathcal{B}} := l_{\mathcal{B}} + r_{\mathcal{B}}$ are the linear representations of the sub-adjacent Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ of the quasi-antiassociative algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Then, the statement that $\mathcal{G}(\mathcal{A}) \bowtie^{-1,\mathrm{ad}_{\mathcal{A}}}_{\mathrm{ad}_{\mathcal{B}}} \mathcal{G}(\mathcal{B})$ is a matched pair of the Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$ follows from Theorem 3.6. By analogous step giving:

$$\mathrm{ad}_{\mathcal{A}}(x)\,[a,b] - [\mathrm{ad}_{\mathcal{A}}(x)a, b] - [a, \mathrm{ad}_{\mathcal{A}}(x)b] - \mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(a)x)b - \mathrm{ad}_{\mathcal{A}}(\mathrm{ad}_{\mathcal{B}}(b)x)a = 0, \tag{3.14}$$

$$\mathrm{ad}_{\mathcal{B}}(a)\,[x,y] - [\mathrm{ad}_{\mathcal{B}}(a)x, y] - [x, \mathrm{ad}_{\mathcal{B}}(a)y]$$

$$-\operatorname{ad}_{\mathcal{B}}(\operatorname{ad}_{\mathcal{A}}(x)a)y-\operatorname{ad}_{\mathcal{B}}(\operatorname{ad}_{\mathcal{A}}(y)a)x=0. \tag{3.15}$$

Thus, we first have:

$$\begin{aligned}
&\operatorname{ad}_{\mathcal{A}}(x)[a,b]-[\operatorname{ad}_{\mathcal{A}}(x)a,b]-[a,\operatorname{ad}_{\mathcal{A}}(x)b]-\operatorname{ad}_{\mathcal{A}}(\operatorname{ad}_{\mathcal{B}}(a)x)b-\operatorname{ad}_{\mathcal{A}}(\operatorname{ad}_{\mathcal{B}}(b)x)a\\
&=(l_{\mathcal{A}}+r_{\mathcal{A}})(x)\,[a,b]-[(l_{\mathcal{A}}+r_{\mathcal{A}})(x)a,b]-[a,(l_{\mathcal{A}}+r_{\mathcal{A}})(x)b]\\
&\quad-(l_{\mathcal{A}}+r_{\mathcal{A}})((l_{\mathcal{B}}+r_{\mathcal{B}})(a)x)b-(l_{\mathcal{A}}+r_{\mathcal{A}})((l_{\mathcal{B}}+r_{\mathcal{B}})(b)x)a\\
&=l_{\mathcal{A}}(x)[a,b]+r_{\mathcal{A}}(x)[a,b]-[l_{\mathcal{A}}(x)a,b]-[r_{\mathcal{A}}(x)a,b]-[a,l_{\mathcal{A}}(x)b]\\
&\quad-[a,r_{\mathcal{A}}(x)b]-l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b-l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b-r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b-r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b\\
&\quad-l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a-r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a\\
&=l_{\mathcal{A}}(x)(a\circ b)+l_{\mathcal{A}}(x)(b\circ a)+r_{\mathcal{A}}(x)(a\circ b)+r_{\mathcal{A}}(x)(b\circ a)-(l_{\mathcal{A}}(x)a)\circ b\\
&\quad-b\circ(l_{\mathcal{A}}(x)a)-(r_{\mathcal{A}}(x)a)\circ b-b\circ(r_{\mathcal{A}}(x)a)-a\circ(l_{\mathcal{A}}(x)b)-(l_{\mathcal{A}}(x)b)\circ a\\
&\quad-a\circ(r_{\mathcal{A}}(x)b)-(r_{\mathcal{A}}(x)b)\circ a-l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b-l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b-r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b\\
&\quad-r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b-l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a-r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a\\
&=\{r_{\mathcal{A}}(x)([a,b])-r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b-a\circ(r_{\mathcal{A}}(x)b)-b\circ(r_{\mathcal{A}}(x)a)\}\\
&\quad+\{-l_{\mathcal{A}}(x)(a\circ b)-l_{\mathcal{A}}(l_{\mathcal{B}}(a)x-r_{\mathcal{B}}(a)x)b-(l_{\mathcal{A}}(x)a-r_{\mathcal{A}}(x)a)\circ b\\
&\quad-r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a-a\circ(l_{\mathcal{A}}(x)b)\}\\
&\quad+\{-l_{\mathcal{A}}(x)(b\circ a)-l_{\mathcal{A}}(l_{\mathcal{B}}(b)x-r_{\mathcal{B}}(b)x)a-(l_{\mathcal{A}}(x)b-r_{\mathcal{A}}(x)b)\circ a\\
&\quad-r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b-b\circ(l_{\mathcal{A}}(x)a)\}=0.
\end{aligned}$$

Secondly:

$$\begin{aligned}
&\operatorname{ad}_{\mathcal{B}}(a)[x,y]-[\operatorname{ad}_{\mathcal{B}}(a)x,y]-[x,\operatorname{ad}_{\mathcal{B}}(a)y]-\operatorname{ad}_{\mathcal{B}}(\operatorname{ad}_{\mathcal{A}}(x)a)y-\operatorname{ad}_{\mathcal{B}}(\operatorname{ad}_{\mathcal{A}}(y)a)x\\
&=(l_{\mathcal{B}}+r_{\mathcal{B}})(a)\,[x,y]-[(l_{\mathcal{B}}+r_{\mathcal{B}})(a)x,y]-[x,(l_{\mathcal{B}}+r_{\mathcal{B}})(a)y]\\
&\quad-(l_{\mathcal{B}}+r_{\mathcal{B}})((l_{\mathcal{A}}+r_{\mathcal{A}})(x)a)y-(l_{\mathcal{B}}+r_{\mathcal{B}})((l_{\mathcal{A}}+r_{\mathcal{A}})(y)a)x\\
&=l_{\mathcal{B}}(a)[x,y]+r_{\mathcal{B}}(a)[x,y]-[l_{\mathcal{B}}(a)x,y]-[r_{\mathcal{B}}(a)x,y]-[x,l_{\mathcal{B}}(a)y]\\
&\quad-[x,r_{\mathcal{B}}(a)y]-l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y-l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y-r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y-r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y\\
&\quad-l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x-l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x-r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x-r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x\\
&=\{r_{\mathcal{A}}(a)[x,y]-r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x-r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y-x(r_{\mathcal{B}}(a)y)-y(r_{\mathcal{B}}(a)x)\}\\
&\quad+\{-l_{\mathcal{B}}(a)(xy)-l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y-(r_{\mathcal{B}}(a)x)y-x(l_{\mathcal{B}}(a)y)-r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x\\
&\quad-(l_{\mathcal{B}}(a)x)y-l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y-l_{\mathcal{B}}(a)(xy)\}+\{-l_{\mathcal{B}}(a)(yx)-l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x\\
&\quad-(r_{\mathcal{B}}(a)y)x-y(l_{\mathcal{B}}(a)x)-r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y-(l_{\mathcal{B}}(a)y)x-l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x\\
&\quad-l_{\mathcal{B}}(a)(yx)\}=0.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\operatorname{ad}_{\mathcal{A}}(x)\,[a,b]-[\operatorname{ad}_{\mathcal{A}}(x)a,b]-[a,\operatorname{ad}_{\mathcal{A}}(x)b]&\\
-\operatorname{ad}_{\mathcal{A}}(\operatorname{ad}_{\mathcal{B}}(a)x)b-\operatorname{ad}_{\mathcal{A}}(\operatorname{ad}_{\mathcal{B}}(b)x)a&=0,
\end{aligned}$$

$$\begin{aligned}
\operatorname{ad}_{\mathcal{B}}(a)\,[x,y]-[\operatorname{ad}_{\mathcal{B}}(a)x,y]-[x,\operatorname{ad}_{\mathcal{B}}(a)y]&\\
-\operatorname{ad}_{\mathcal{B}}(\operatorname{ad}_{\mathcal{A}}(x)a)y-\operatorname{ad}_{\mathcal{B}}(\operatorname{ad}_{\mathcal{A}}(y)a)x&=0.
\end{aligned}$$

Hence, $(\mathcal{G}(\mathcal{A}),\mathcal{G}(\mathcal{B}),\operatorname{ad}_{\mathcal{A}},\operatorname{ad}_{\mathcal{B}})$ is a matched pair of sub-adjacent Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{B})$. $\square$

Definition 3.8. Let (l, r, V) be a bimodule of a quasi-antiassociative algebra $\mathcal{A}$, where V is a finite dimensional vector space. The dual maps l^*, r^* of the linear maps l, r, are defined, respectively, as: $l^*, r^* : \mathcal{A} \to \mathfrak{gl}(V^*)$ such that: for all $x \in \mathcal{A}, u^* \in V^*, v \in V$,

$$\begin{array}{rcl} l^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*) \\ x & \longmapsto & l^*_x : \begin{array}{rcl} V^* & \longrightarrow & V^* \\ u^* & \longmapsto & l^*_x u^* : \begin{array}{rcl} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle l^*_x u^*, v\rangle := \langle u^*, l_x v\rangle , \end{array} \end{array} \end{array} \tag{3.16}$$

$$\begin{array}{rcl} r^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*) \\ x & \longmapsto & r^*_x : \begin{array}{rcl} V^* & \longrightarrow & V^* \\ u^* & \longmapsto & r^*_x u^* : \begin{array}{rcl} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle r^*_x u^*, v\rangle := \langle u^*, -r_x v\rangle . \end{array} \end{array} \end{array} \tag{3.17}$$

Proposition 3.9. *Let $(\mathcal{A}, \cdot)$ be a quasi-antiassociative algebra and l, r be two linear maps $\mathcal{A} \to \mathfrak{gl}(V)$, where V is a finite dimensional vector space. The following conditions are equivalent:*

1. (l, r, V) *is a bimodule of* $\mathcal{A}$.
2. $(l^* + r^*, -r^*, V^*)$ *is a bimodule of* $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (2): Suppose that (l, r, V) is a bimodule of $(\mathcal{A}, \cdot)$ and show that $(l^* + r^*, -r^*, V^*)$ is also a bimodule of $(\mathcal{A}, \cdot)$. We have: firstly,

$$\begin{aligned} \langle[(l^*+r^*)_{xy} + (l^*+r^*)_x(l^*+r^*)_y]u^*, v\rangle &= \langle(l^*+r^*)_{xy}u^*, v\rangle + \langle[(l^*+r^*)_x(l^*+r^*)_y]u^*, v\rangle \\ &= -\langle(l+r)_{xy}(v), u^*\rangle - \langle[(l+r)_y(l+r)_x](v), u^*\rangle \\ &= -\langle(l^*+r^*)_{yx}u^*, v\rangle - \langle(l^*+r^*)_y(l^*+r^*)_x)u^*, v\rangle. \end{aligned}$$

Therefore,

$$(l^*+r^*)_{xy} + (l^*+r^*)_x(l^*+r^*)_y = -(l^*+r^*)_{yx} - (l^*+r^*)_y(l^*+r^*)_x,\ \forall\, x, y \in \mathcal{A} \tag{3.18}$$

Secondly,

$$\begin{aligned} \big\langle r^*_y(l^*+r^*)_x u^* + (l^*+r^*)_x r^*_y u^*, v\big\rangle &= \langle(l+r)_x r_y v, u^*\rangle + \langle r_y(l+r)_x v, u^*\rangle \\ &= \langle l_x(r_y(v)) + r_x(r_y(v)), u^*\rangle + \langle r_y(l_x(v)) + r_y(r_x(v)), u^*\rangle \\ &= \langle r_x(r_y(v)) + r_{xy}v, u^*\rangle \\ &= \big\langle(r^*_y r^*_x + r^*_{xy})u^*, v\big\rangle. \end{aligned}$$

Therefore

$$(-r^*)_y(-r^*)_x + (-r^*)_{xy} = -[(l^*+r^*)_y, -r^*_x],\ \forall\, x, y\ \in \mathcal{A}. \tag{3.19}$$

By considering the relations (3.18) and (3.19), we conclude that $(l^*+r^*, -r^*, V^*)$ is a bimodule of $(\mathcal{A}, \cdot)$.

(2) $\Rightarrow$ (1): The converse, (i.e., by supposing that $(l^*+r^*, -r^*, V^*)$ is a bimodule of $(\mathcal{A}, \cdot)$ then (l, r, V) is also a bimodule of $(\mathcal{A}, \cdot)$), can be proved by direct calculations by using similar relations as for the first part of the proof. □

Example 3.10. Let $\mathcal{A}$ be a quasi-antiassociative algebra. Then $(L, 0)$, $(\mathrm{ad}, 0)$ and (L, R) are bimodules of $\mathcal{A}$, where $\mathrm{ad} = L+R : \mathcal{A} \to gl(\mathcal{A})$ is the adjoint representation of the Mock Lie algebra $\mathcal{G}(\mathcal{A})$. On the other hand, $(L^*, 0)$, $(\mathrm{ad}^*, 0)$ and $(\mathrm{ad}^*, -R^*)$ are bimodules of $\mathcal{A}$, too, where ad^* and L^* are the dual representations of the adjoint representation and the regular representation of the sub-adjacent Mock Lie algebra $\mathcal{G}(\mathcal{A})$ respectively.

Theorem 3.11. *Let $(\mathcal{A}, \cdot)$ be a quasi-antiassociative algebra. Suppose that there exists a quasi-antiassociative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. Then, $(\mathcal{A}, \mathcal{A}^*, \mathrm{ad}^*_{\cdot}, -R^*_{\cdot}, \mathrm{ad}^*_{\circ}, -R^*_{\circ})$ is a matched pair of quasi-antiassociative algebras $\mathcal{A}$ and $\mathcal{A}^*$ if and only if $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), L^*_{\cdot}, L^*_{\circ})$ is a matched pair of Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$.*

Proof. By considering the Corollary 3.7, setting $l_{\mathcal{A}} := \mathrm{ad}^*_{\cdot} = L^*_{\cdot} + R^*_{\cdot}, r_{\mathcal{A}} := -R^*_{\cdot}, l_{\mathcal{B}} := \mathrm{ad}^*_{\circ} = L^*_{\circ} + R^*_{\circ}, r_{\mathcal{B}} := -R^*_{\circ}$, and exploiting the Definition 3.5 with $\mathcal{G} := \mathcal{G}(\mathcal{A}), \mathcal{H} := \mathcal{G}(\mathcal{A}^*), \rho := L^*_{\cdot}, \mu := L^*_{\circ}$, the proof is established. □

4. Quasi-antiassociative bialgebras

Definition 4.1. [2] Let $(\mathcal{G}, [,])$ be a Mock-Lie algebra. A 1-cocycle δ is associated to a representation (ρ, V) of $(\mathcal{G}, [,])$, if the equation

$$\delta([x, y]) = -\rho(x)\delta(y) - \rho(y)\delta(x), \ \ \forall x, y \in \mathcal{G}, \tag{4.1}$$

is satisfied.

Proposition 4.2. *Let $(\mathcal{G}, [,])$ be a Mock-Lie algebra. Suppose $\rho : \mathcal{G} \to \mathfrak{gl}(V)$ and $\mu : \mathcal{G} \to \mathfrak{gl}(W)$ be two linear representations of $\mathcal{G}$, where V and W are two vector spaces. Then, the linear map $\rho \otimes 1 + 1 \otimes \mu : \mathcal{G} \to \mathfrak{gl}(V \otimes W)$ given by $(\rho \otimes 1 + 1 \otimes \mu)(x)(v, w) := \rho(x)v \otimes w + v \otimes \mu(x)w$ is also a representation of $\mathcal{G}$.*

Proof. Let $(\mathcal{G}, [,])$ be a Mock-Lie algebra, $(\mathcal{G}, \rho)$ and $(\mathcal{G}, \mu)$ be two representations of $(\mathcal{G}, [,])$. For all $x, y \in \mathcal{G}$ and $v \otimes w \in V \otimes W$, we have

$$(\rho \otimes 1 + 1 \otimes \mu)([x, y]) = \rho([x, y]) \otimes 1 + 1 \otimes \mu([x, y])$$

$$
\begin{aligned}
&= -\rho(x)\rho(y) \otimes 1 - 1 \otimes \mu(x)\mu(y) \\
&\quad -1 \otimes \mu(y)\mu(x) - \rho(y)\rho(x) \otimes 1 \\
&= -\rho(x)\rho(y)v \otimes 1 - 1 \otimes \mu(x)\mu(y) \\
&\quad -1 \otimes \mu(y)\mu(x) - \rho(y)\rho(x) \otimes 1 \\
&\quad -\rho(x) \otimes \mu(y) - \rho(x) \otimes \mu(y) \\
&\quad -\rho(y) \otimes \mu(x) - \rho(y) \otimes \mu(x) \\
&= -[\rho(x) \otimes 1, \rho(y) \otimes 1] - [\rho(x) \otimes 1, 1 \otimes \mu(y)] \\
&\quad -[1 \otimes \mu(x), \rho(y) \otimes 1] - [1 \otimes \mu(x), \rho(y) \otimes 1] \\
&= -[\rho(x) \otimes 1 + 1 \otimes \mu(x), \rho(y) \otimes 1 + 1 \otimes \mu(y)].
\end{aligned}
$$

Therefore, we have $\forall x, y \in \mathcal{G}$;

$$(\rho \otimes 1 + 1 \otimes \mu)([x, y]) = -[\rho(x) \otimes 1 + 1 \otimes \mu(x), \rho(y) \otimes 1 + 1 \otimes \mu(y)],$$

which exactly means that $\rho \otimes 1 + 1 \otimes \mu$ is a representation of $(\mathcal{G}; [,])$ on $V \otimes W$. $\square$

Theorem 4.3. *Let $\mathcal{A}$ be a quasi-antiassociative algebra with the product given by the linear map $\beta^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Suppose there is a quasi-antiassociative algebra structure " $\circ$ " on the dual space $\mathcal{A}^*$ provided by a linear map $\alpha^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$. Then, $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), L^*_{\cdot}, L^*_{\circ})$ is a matched pair of Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$ if and only if $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is an antiderivation of $\mathcal{G}(\mathcal{A})$ associated to $. \otimes 1 + 1 \otimes \mathrm{ad}.$ and $\beta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ is an antiderivation of $\mathcal{G}(\mathcal{A}^*)$ associated to $L_\circ \otimes 1 + 1 \otimes \mathrm{ad}_\circ$.*

Proof. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of $\mathcal{A}$ and $\{e_1^*, e_2^*, \cdots, e_n^*\}$ its dual basis. Consider $e_i \cdot e_j = \sum_{k=1}^{n} c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^{n} f_{ij}^k e_k^*$, where $c_{ij}^k, f_{ij}^k \in \mathbb{K}$ are structure constants associated to $\cdot$ and $\circ$, respectively. Then, it follows that: $\alpha(e_k) = \sum_{i,j=1}^{n} f_{ij}^k e_i \otimes e_j, \quad \beta(e_k^*) = \sum_{i,j=1}^{n} c_{ij}^k e_i^* \otimes e_j^*,$ and

$$\alpha([e_i, e_j]) = \sum_{m,l=1}^{n} \sum_{k=1}^{n} \left\{ (c_{ij}^k + c_{ji}^k) f_{ml}^k \right\} e_m \otimes e_l, \tag{4.2}$$

$$\beta([e_i^*, e_j^*]) = \sum_{m,l=1}^{n} \sum_{k=1}^{n} \left\{ (f_{ij}^k + f_{ji}^k) c_{ml}^k \right\} e_m^* \otimes e_l^*. \tag{4.3}$$

Now, let us compute the quantities:

$\{(-L.)(e_i) \otimes 1 + 1 \otimes (-\mathrm{ad}.)(e_i)\}\alpha(e_j) + \{(-L.)(e_j) \otimes 1 + 1 \otimes (-\mathrm{ad}.)(e_j)\}\alpha(e_i)$

and

$\{(-L_\circ)(e_i^*)\otimes 1+1\otimes(-\operatorname{ad}_\circ)(e_i^*)\}\beta(e_j^*)+\{(-L_\circ)(e_j^*)\otimes 1+1\otimes(-\operatorname{ad}_\circ)(e_j^*)\}\beta(e_i^*)$.

We have:

$$\begin{aligned}
&\{(-L.)(e_i)\otimes 1+1\otimes(-\mathrm{ad}.)(e_i)\}\alpha(e_j)+\{(-L.)(e_j)\otimes 1+1\otimes(-\mathrm{ad}.)(e_j)\}\alpha(e_i)\\
&=\{-(L.(e_i)\otimes 1)\alpha(e_j)-(L.(e_j)\otimes 1)\alpha(e_i)\}+\{-(1\otimes\mathrm{ad}.(e_j))\alpha(e_i)\\
&-(1\otimes\mathrm{ad}.(e_i))\alpha(e_j)\}=\sum_{k,l=1}^{n}\{-f_{kl}^{j}(L.(e_i)\otimes 1)(e_k\otimes e_l)-f_{kl}^{i}(L.(e_j)\otimes 1)(e_k\otimes e_l)\}\\
&+\sum_{m,k=1}^{n}\{-f_{mk}^{j}(1\otimes\mathrm{ad}.(e_i))(e_m\otimes e_k)-f_{mk}^{i}(1\otimes\mathrm{ad}.(e_j))(e_m\otimes e_k)\}\\
&=\sum_{k,l=1}^{n}\{-f_{kl}^{j}([e_i,e_k]\otimes e_l)-f_{kl}^{i}([e_j,e_k]\otimes e_l)\}\\
&+\sum_{m,k=1}^{n}\{-f_{mk}^{j}(e_m\otimes[e_i,e_k])-f_{mk}^{i}(e_m\otimes[e_j,e_k])\}\\
&=\sum_{k,l=1}^{n}\left\{-f_{kl}^{j}\left(\sum_{m=1}^{n}(c_{ik}^{m}+c_{ki}^{m})(e_m\otimes e_l)\right)-f_{kl}^{i}\left(\sum_{m=1}^{n}(c_{jk}^{m}+c_{kj}^{m})(e_m\otimes e_l)\right)\right\}\\
&+\sum_{m,k=1}^{n}\left\{-f_{mk}^{j}\left(\sum_{l=1}^{n}(c_{ik}^{l}+c_{ki}^{l})(e_m\otimes e_l)\right)-f_{mk}^{i}\left(\sum_{l=1}^{n}(c_{jk}^{l}+c_{kj}^{l})(e_m\otimes e_l)\right)\right\}\\
&=\sum_{m,l=1}^{n}\sum_{k=1}^{n}\left\{-f_{kl}^{j}(c_{ik}^{m}+c_{ki}^{m})-f_{kl}^{i}(c_{jk}^{m}+c_{kj}^{m})-f_{mk}^{j}(c_{ik}^{l}+c_{ki}^{l})-f_{mk}^{i}(c_{jk}^{l}+c_{kj}^{l})\right\}(e_m\otimes e_l).
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
&\{(-L.)(e_i)\otimes 1+1\otimes(-\mathrm{ad}.)(e_i)\}\alpha(e_j)+\{(-L.)(e_j)\otimes 1+1\otimes(-\mathrm{ad}.)(e_j)\}\alpha(e_i)\\
&=\sum_{m,l=1}^{n}\sum_{k=1}^{n}\{-f_{kl}^{j}(c_{ik}^{m}+c_{ki}^{m})-f_{kl}^{i}(c_{jk}^{m}+c_{kj}^{m})-f_{mk}^{j}(c_{ik}^{l}+c_{ki}^{l}) \qquad (4.4)\\
&-f_{mk}^{i}(c_{jk}^{l}+c_{kj}^{l})\}e_m\otimes e_l.
\end{aligned}$$

Taking into account the fact that α is an antiderivation of $\mathcal{G}(\mathcal{A})$ associated to $L.\otimes 1+1\otimes\mathrm{ad}.$, and using the relations (4.2) and (4.4) yield:

$$\sum_{k=1}^{n}(c_{ij}^{k}+c_{ji}^{k})f_{ml}^{k}=\sum_{k=1}^{n}\{-f_{kl}^{i}(c_{jk}^{m}+c_{kj}^{m})-f_{kl}^{j}(c_{ik}^{m}+c_{ki}^{m})-f_{mk}^{i}(c_{jk}^{l}+c_{kj}^{l})-f_{mk}^{j}(c_{ik}^{l}+c_{ki}^{l})\}. \quad (4.5)$$

Besides, we obtain:

$$\{(-L_\circ)(e_i^*)\otimes 1+1\otimes(-\operatorname{ad}_\circ)(e_i^*)\}\beta(e_j^*)+\{(-L_\circ)(e_j^*)\otimes 1+1\otimes(-\operatorname{ad}_\circ)(e_j^*)\}\beta(e_i^*)$$

$$
\begin{aligned}
&= \{(-L_\circ)(e_i^*)\otimes 1)\beta(e_j^*) + (-L_\circ)(e_j^*\otimes 1)\beta(e_i^*)\} \\
&\quad +\{(1\otimes(-\operatorname{ad}_\circ)(e_i^*))\beta(e_j^*) + (1\otimes(-\operatorname{ad}_\circ)(e_j^*))\beta(e_i^*)\} \\
&= \sum_{k,l=1}^{n}\{-c_{kl}^{j}(L_\circ(e_i^*)\otimes 1) - c_{kl}^{i}(L_\circ(e_j^*)\otimes 1)\}(e_k^*\otimes e_l^*) \\
&\quad + \sum_{m,k=1}^{n}\{-c_{mk}^{j}(1\otimes\operatorname{ad}_\circ(e_i^*)) - c_{mk}^{i}(1\otimes\operatorname{ad}_\circ(e_j^*))\}(e_m^*\otimes e_k^*) \\
&= \sum_{k,l=1}^{n}\Big\{-c_{kl}^{j}[e_i^*,e_k^*] - c_{kl}^{i}[e_j^*,e_k^*]\Big\}\otimes e_l^* + \sum_{m,k=1}^{n} e_m^*\otimes\Big\{-c_{mk}^{j}[e_i^*,e_k^*] - c_{mk}^{i}[e_j^*,e_k^*]\Big\} \\
&= \sum_{k,l=1}^{n}\sum_{m=1}^{n}\Big\{-c_{kl}^{j}(f_{ik}^{m}+f_{ki}^{m}) - c_{kl}^{i}(f_{jk}^{m}+f_{kj}^{m})\Big\}(e_m^*\otimes e_l^*) \\
&\quad + \sum_{m,k=1}^{n}\sum_{l=1}^{n}\Big\{-c_{mk}^{j}(f_{ik}^{l}+f_{ki}^{l}) - c_{mk}^{i}(f_{jk}^{l}+f_{kj}^{l})\Big\}(e_m^*\otimes e_l^*) \\
&= \sum_{m,l=1}^{n}\sum_{k=1}^{n}\Big\{-c_{kl}^{j}(f_{ik}^{m}+f_{ki}^{m}) - c_{kl}^{i}(f_{jk}^{m}+f_{kj}^{m}) - c_{mk}^{j}(f_{ik}^{l}+f_{ki}^{l}) - c_{mk}^{i}(f_{jk}^{l}+f_{kj}^{l})\Big\}(e_m^*\otimes e_l^*).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\{(-L_\circ)(e_i^*)\otimes 1+1\otimes(-\operatorname{ad}_\circ)(e_i^*)\}\beta(e_j^*)+\{(-L_\circ)(e_j^*)\otimes 1+1\otimes(-\operatorname{ad}_\circ)(e_j^*)\}\beta(e_i^*) \\
&=\sum_{m,l=1}^{n}\sum_{k=1}^{n}\{-c_{kl}^{j}(f_{ik}^{m}+f_{ki}^{m})-c_{kl}^{i}(f_{jk}^{m}+f_{kj}^{m})-c_{mk}^{j}(f_{ik}^{l}+f_{ki}^{l})-c_{mk}^{i}(f_{jk}^{l}+f_{kj}^{l})\}(e_m^*\otimes e_l^*). \quad (4.6)
\end{aligned}
$$

Since β is the antiderivation issued from $L_\circ\otimes 1+1\otimes\operatorname{ad}_\circ$, using the relations (4.3) and (4.6) we obtain:

$$
\sum_{k=1}^{n}(f_{ij}^{k}+f_{ji}^{k})c_{ml}^{k} =\sum_{k=1}^{n}\{-c_{kl}^{i}(f_{jk}^{m}+f_{kj}^{m})-c_{kl}^{j}(f_{ik}^{m}+f_{ki}^{m})-c_{mk}^{i}(f_{jk}^{l}+f_{kj}^{l})-c_{mk}^{j}(f_{ik}^{l}+f_{ki}^{l})\}. \quad (4.7)
$$

Now, let us find the relations associated to the equations (3.3)-(3.4) of the matched pair of Mock-Lie algebras $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathcal{A}^*)$. We have $\forall\, i,j,k$:

$$
\langle(-L_\cdot^*)(e_i)e_j^*, e_k\rangle=-\langle e_j^*,L_\cdot(e_i)e_k\rangle=-\langle e_j^*,[e_i,e_k]\rangle=-\left\langle e_j^*,\sum_{m=1}^{n}(c_{ik}^{m}+c_{ki}^{m})e_m\right\rangle=-(c_{ik}^{j}+c_{ki}^{j}),
$$

$$
\langle(-L_\cdot^*)(e_i)e_j^*, e_k\rangle = -\left\langle\sum_{k=1}^{m}(c_{ik}^{j}+c_{ki}^{j})e_j^*, e_k\right\rangle,
$$

providing

$$
(-L_\cdot^*)(e_i)e_j^* = -\sum_{k=1}^{n}(c_{ik}^{j}+c_{ki}^{j})e_k^*.
$$

Similarly,

$$
(-L_\circ^*)(e_i^*)e_j = -\sum_{k=1}^{n}(f_{ik}^{j}+f_{ki}^{j})e_k,
$$

$$
(-L_\circ^*)(e_m^*)[e_i,e_j] = \sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})(-L_\circ^*)(e_m^*)e_k - \sum_{l=1}^{n}\sum_{k=1}^{n}(c_{ij}^{k}-c_{ji}^{k})(f_{ml}^{k}-f_{lm}^{k})e_l,
$$

and,

$$(-L^*_\circ)(e^*_m)[e_i,e_j] = -\sum_{l=1}^{n}\sum_{k=1}^{n}(c^k_{ij}+c^k_{ji})(f^k_{ml}+f^k_{lm})e_l.$$

Then

$$\begin{aligned}
&- L^*_\circ(L^*_\cdot(e_i)e^*_m)e_j+[e_i,L^*_\circ(e^*_m)e_j] - L^*_\circ(L^*_\cdot(e_j)e^*_m)e_i+[L^*_\circ(e^*_m)e_i,e_j]\\
&-\sum_{k=1}^{n}(c^m_{ik}+c^m_{ki})L^*_\circ(e^*_k)e_j - \sum_{k=1}^{n}(f^j_{mk}+f^j_{km})[e_i,e_k]\\
&-\sum_{k=1}^{n}(c^m_{jk}+c^m_{kj})L^*_\circ(e^*_k)e_i - \sum_{k=1}^{n}(f^i_{mk}+f^i_{km})[e_k,e_j]\\
&= -\sum_{k=1}^{n}\sum_{l=1}^{n}\Big\{(c^m_{ik}+c^m_{ki})(f^j_{kl}+f^j_{lk})e_l+(f^j_{mk}+f^j_{km})(c^l_{ik}+c^l_{ki})e_l\Big\}\\
&-\sum_{k=1}^{n}\sum_{l=1}^{n}\Big\{(c^m_{jk}+c^m_{kj})(f^i_{kl}+f^i_{lk})e_l+(f^i_{mk}+f^i_{km})(c^l_{kj}+c^l_{jk})e_l\Big\}\\
&= -\sum_{l=1}^{n}\sum_{k=1}^{n}\{(c^m_{ik}+c^m_{ki})(f^j_{kl}+f^j_{lk})+(f^j_{mk}+f^j_{km})(c^l_{ik}+c^l_{ki})\\
&+ (c^m_{jk}+c^m_{kj})(f^i_{kl}+f^i_{lk})+(f^i_{mk}+f^i_{km})(c^l_{kj}+c^l_{jk})\}e_l.
\end{aligned}$$

Using the fact that $(\mathcal{G}(\mathcal{A}),\mathcal{G}(\mathcal{A}^*),L^*_\cdot,L^*_\circ)$ is a matched pair of Mock-Lie algebras, we have

$$\begin{aligned}
\sum_{k=1}^{n}(c^k_{ij}+c^k_{ji})(f^k_{ml}+f^k_{lm}) = -\sum_{k=1}^{n}&\{(c^m_{ik}+c^m_{ki})(f^j_{kl}+f^j_{lk})+(f^j_{mk}+f^j_{km})(c^l_{ik}+c^l_{ki})\\
&+ (c^m_{jk}+c^m_{kj})(f^i_{kl}+f^i_{lk})+(f^i_{mk}+f^i_{km})(c^l_{jk}+c^l_{kj})\},
\end{aligned}$$

that is,

$$\begin{aligned}
&\sum_{k=1}^{n}(c^k_{ij}+c^k_{ji})f^k_{ml}+\sum_{k=1}^{n}(c^m_{ik}+c^m_{ki})f^j_{kl}+(c^l_{ik}+c^l_{ki})f^j_{mk}+(c^m_{jk}+c^m_{kj})f^i_{kl}+(c^l_{jk}+c^l_{kj})f^i_{mk}\\
&= -\sum_{k=1}^{n}(c^k_{ij}+c^k_{ji})f^k_{lm} - \sum_{k=1}^{n}(c^m_{ik}+c^m_{ki})f^j_{lk} - (c^l_{ik}+c^l_{ki})f^j_{km} - (c^m_{jk}+c^m_{kj})f^i_{lk} - (c^l_{jk}+c^l_{kj})f^i_{km}.
\end{aligned}$$

Replacing l (resp. m) by m (resp. l) in the right-hand side of the equality leads to:

$$\sum_{k=1}^{n}(c^k_{ij}+c^k_{ji})f^k_{ml} =\sum_{k=1}^{n}\{-(c^m_{ik}+c^m_{ki})f^j_{kl}-(c^l_{ik}+c^l_{ki})f^j_{mk}-(c^m_{jk}+c^m_{kj})f^i_{kl}-(c^l_{jk}+c^l_{kj})f^i_{mk}\},$$

which is identical to the equation (4.5). Besides,

$$(-\operatorname{ad}^*_\cdot)(e_m)[e^*_i,e^*_j] = -\sum_{l=1}^{n}\sum_{k=1}^{n}\Big\{(f^k_{ij}+f^k_{ji})(c^k_{ml}+c^k_{lm})\Big\}\, e^*_l$$

$$\begin{aligned}
&-\operatorname{ad}^*_\cdot(\operatorname{ad}^*_\circ(e^*_i)e_m)e^*_j+[e^*_i,\operatorname{ad}^*_\cdot(e_m)e^*_j]-\operatorname{ad}^*_\cdot(\operatorname{ad}^*_\circ(e^*_j)e_m)e^*_m+[\operatorname{ad}^*_\cdot(e_m)e^*_i,e^*_j]\\
&-(\operatorname{ad}^*_\cdot)(e_m)[e^*_i,e^*_j]
\end{aligned}$$

$$= \sum_{l=1}^{n}\sum_{k=1}^{n}\{-(f_{ik}^m+f_{ki}^m)(c_{kl}^j+c_{lk}^j)-(c_{mk}^j+c_{km}^j)(f_{ik}^l+f_{ki}^l)-(f_{jk}^m+f_{kj}^m)(c_{kl}^i+c_{lk}^i)$$
$$-(c_{mk}^i+c_{km}^i)(f_{kj}^l+f_{jk}^l)\}e_l^*.$$

Then, with $\mathcal{G}(\mathcal{A}) \bowtie_{-L_\circ^*}^{-1,-L_\cdot^*} \mathcal{G}(\mathcal{A}^*)$ and the above relation gives

$$\sum_{k=1}^{n}(f_{ij}^k+f_{ji}^k)(c_{ml}^k+c_{lm}^k) = \sum_{k=1}^{n} -(f_{ik}^m+f_{ki}^m)(c_{kl}^j+c_{lk}^j) - (c_{mk}^j+c_{km}^j)(f_{ik}^l+f_{ki}^l)$$
$$-(f_{jk}^m+f_{kj}^m)(c_{kl}^i+c_{lk}^i) - (c_{mk}^i+c_{km}^i)(f_{jk}^l+f_{kj}^l),$$

i.e.

$$\sum_{k=1}^{n}(f_{ij}^k+f_{ji}^k)c_{ml}^k+\sum_{k=1}^{n}c_{kl}^j(f_{ik}^m+f_{ki}^m)+c_{kl}^i(f_{jk}^m+f_{kj}^m)+c_{mk}^j(f_{ik}^l+f_{ki}^l)-c_{mk}^i(f_{jk}^l+f_{kj}^l)$$
$$=\sum_{k=1}^{n}-(f_{ij}^k+f_{ji}^k)c_{lm}^k-\sum_{k=1}^{n}c_{lk}^j(f_{ik}^m+f_{ki}^m)-c_{lk}^i(f_{jk}^m+f_{kj}^m)-c_{km}^j(f_{ik}^l+f_{ki}^l)-c_{km}^i(f_{jk}^l+f_{kj}^l).$$

Replacing l, (resp. m), by m,(resp. l), in the right-hand side of the equality leads to

$$\sum_{k=1}^{n}(f_{ij}^k+f_{ji}^k)c_{ml}^k=\sum_{k=1}^{n}-c_{kl}^j(f_{ik}^m+f_{ki}^m)-c_{kl}^i(f_{jk}^m+f_{kj}^m)-c_{mk}^j(f_{ik}^l+f_{ki}^l)-c_{mk}^i(f_{kj}^l+f_{jk}^l),$$

which is identical to the equation (4.7). □

Definition 4.4. Let $\mathcal{A}$ be a vector space. A *quasi-antiassociative bialgebra* structure on $\mathcal{A}$ is a pair of linear maps (α, β) such that $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\beta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ satisfying:

1. $\alpha^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ is a quasi-antiassociative algebra structure on $\mathcal{A}^*$,
2. $\beta^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a quasi-antiassociative algebra structure on $\mathcal{A}$,
3. $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a 1-cocycle of $\mathcal{G}(\mathcal{A})$ associated to $L_\cdot \otimes 1 + 1 \otimes \mathrm{ad}_\cdot$,
4. $\beta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ is a 1-cocycle of $\mathcal{G}(\mathcal{A}^*)$ associated to $L_\circ \otimes 1+1 \otimes \mathrm{ad}_\circ$.

We also denote this quasi-antiassociative bialgebra by $(\mathcal{A}, \mathcal{A}^*, \alpha, \beta)$ or simply $(\mathcal{A}, \mathcal{A}^*)$.

Remark 4.5. A quasi-antiassociative bialgebra can also be called a Mock-pre-Lie bialgebra or a pre-Jacobi-Jordan bialgebra.

Proposition 4.6. *Let $(\mathcal{A}, \mathcal{A}^*, \alpha, \beta)$ be a quasi-antiassociative bialgebra. Then its dual $(\mathcal{A}^*, \mathcal{A}, \beta, \alpha)$ is also a quasi-antiassociative bialgebra.*

Proposition 4.7. *Let $(\mathcal{A}, \cdot)$ be a quasi-antiassociative algebra and $(\mathcal{A}^*, \circ)$ be a quasi-antiassociative algebra structure on its dual space $\mathcal{A}^*$. Then the following conditions are equivalent:*

1. $(\mathcal{A}, \mathcal{A}^*, \mathrm{ad}^*_\cdot, -R^*_\cdot, \mathrm{ad}^*_\circ, -R^*_\circ)$ *is a matched pair of quasi-antiassociative algebras;*
2. $(\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*), L^*_\cdot, L^*_\circ)$ *is a matched pair of sub-adjacent Mock-Lie algebras;*
3. $(\mathcal{A}, \mathcal{A}^*)$ *is a quasi-antiassociative bialgebra.*

Definition 4.8. Let $(\mathcal{A}, \mathcal{A}^*, \alpha_\mathcal{A}, \beta_\mathcal{A})$ and $(\mathcal{H}, \mathcal{H}^*, \alpha_\mathcal{H}, \beta_\mathcal{H})$ be two quasi-antiassociative bialgebras. A homomorphism of quasi-antiassociative bialgebras $\varphi : \mathcal{A} \to \mathcal{H}$ is a homomorphism of quasi-antiassociative algebras such that $\varphi^* : \mathcal{H}^* \to \mathcal{A}^*$ is also a homomorphism of quasi-antiassociative algebras, that is, φ satisfies

$$(\varphi\otimes\varphi)\alpha_\mathcal{A}(x) = \alpha_\mathcal{H}(\varphi(x)), \quad (\varphi^*\otimes\varphi^*)\beta_\mathcal{H}(a^*) = \beta_\mathcal{A}(\varphi^*(a^*)), \forall\, x \in \mathcal{A}, a^* \in \mathcal{H}^*.$$

An isomorphism of quasi-antiassociative bialgebras is an invertible homomorphism of quasi-antiassociative bialgebras.

Proposition 4.9. *Two matched pairs of Mock-Lie algebras are isomorphic if and only if their corresponding quasi-antiassociative bialgebras are isomorphic.*

Proof. Let $(\mathcal{G}(\mathcal{A}) \bowtie \mathcal{G}(\mathcal{A}^*), \mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*))$ and $(\mathcal{G}(\mathcal{H}) \bowtie \mathcal{G}(\mathcal{H}^*), \mathcal{G}(\mathcal{H}), \mathcal{G}(\mathcal{H}^*))$ be two matched pairs of Mock-Lie algebras. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathcal{A}$ and $\{e_1^*, \cdots, e_n^*\}$ be its dual basis. If $\varphi : \mathcal{G}(\mathcal{A}) \bowtie \mathcal{G}(\mathcal{A}^*) \to \mathcal{G}(\mathcal{H}) \bowtie \mathcal{G}(\mathcal{H}^*)$ is an isomorphism of matched pairs of Mock-Lie algebras, then $\varphi|_\mathcal{A} : \mathcal{A} \to \mathcal{H}$ and $\varphi|_{\mathcal{A}^*} : \mathcal{A}^* \to \mathcal{H}^*$ are isomorphisms of quasi-antiassociative algebras. Moreover, $\varphi|_{\mathcal{A}^*} = (\varphi|_\mathcal{A})^{*-1}$. Hence $(\mathcal{A}, \mathcal{A}^*)$ and $(\mathcal{H}, \mathcal{H}^*)$ are isomorphic as quasi-antiassociative bialgebras. On the other hand, let $(\mathcal{A}, \mathcal{A}^*)$ and $(\mathcal{H}, \mathcal{H}^*)$ be two isomorphic quasi-antiassociative bialgebras and $\varphi' : \mathcal{A} \to \mathcal{H}$ be an isomorphism of quasi-antiassociative bialgebras. Set $\varphi : \mathcal{A} \oplus \mathcal{A}^* \to \mathcal{H} \oplus \mathcal{H}^*$ be a linear map given by

$$\varphi(x) = \varphi'(x), \quad \varphi(a^*) = (\varphi'^*)^{-1}(a^*), \quad \forall x \in \mathcal{A}, a^* \in \mathcal{A}^*.$$

Then it is easy to know that φ is an isomorphism of the two matched pairs of Mock-Lie algebras $(\mathcal{G}(\mathcal{A}) \bowtie \mathcal{G}(\mathcal{A}^*), \mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{A}^*))$ and $(\mathcal{G}(\mathcal{H}) \bowtie \mathcal{G}(\mathcal{H}^*), \mathcal{G}(\mathcal{H}), \mathcal{G}(\mathcal{H}^*))$. □

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