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Magnifiers in some semigroups of partial transformations

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Abstract. Let $P(X)$ be the semigroup of all partial transformations on a set X. For a nonempty subset Y of X and a submonoid $\mathbb{S}(Y)$ of $P(Y)$, denote by $P_{\mathbb{S}(Y)}(X)$ the semigroup of all partial transformations $f \in P(X)$ such that the restriction $f|_Y$ of f to Y belongs to $\mathbb{S}(Y)$. We describe left and right magnifiers in $P_{\mathbb{S}(Y)}(X)$. We apply these general results to describe magnifiers in $P_{\mathcal{I}(Y)}(X)$, where $\mathcal{I}(Y)$ is the symmetric inverse semigroup on Y. We identify corresponding known results on $P_{P(Y)}(X)$ and $P_{\Omega(Y)}(X)$, where $\Omega(Y)$ is the submonoid of $P(Y)$ consisting of all partial surjective transformations on Y.

1. Introduction

In 1963, Ljapin introduced the notions of left and right magnifiers in a semigroup [8]. For a given semigroup S, an element $s \in S$ is a *left* (*right*) *magnifier* in S if S contains a proper subset M such that $sM = S$ ($Ms = S$). Various properties of semigroups containing left or right magnifiers have been widely investigated (see, [1, 3, 4, 10, 11]).

Let X be a set. Denote by $P(X)$ the set of all partial transformations on X. It is well-known that $P(X)$ is a semigroup under the composition [5, p. 41, Exercise 12]. Studies on left and right magnifiers in various subsemigroups of $P(X)$ have been recently intensified (see, [2, 6, 9, 12, 13]).

For a subset Y of a set X and a subsemigroup $\mathbb{S}(Y)$ of $P(Y)$, let $P_{\mathbb{S}(Y)}(X)$ denote the semigroup under the composition consisting of all partial transformations $f \in P(X)$ such that the restriction $f|_Y$ of f to Y belongs to $\mathbb{S}(Y)$. Using symbols,

$$P_{\mathbb{S}(Y)}(X) := \big\{f \in P(X)\colon f|_Y \in \mathbb{S}(Y)\big\}.$$

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Semigroup $P_{\mathbb{S}(Y)}(X)$ was introduced by Konieczny in [7, p.124]. Notice that if $\mathbb{S}(Y)$ contains the identity transformation on Y, then $P_{\mathbb{S}(Y)}(X)$ contains the identity transformation on X. Semigroup $P_{\mathbb{S}(Y)}(X)$ generalizes the semigroup $\{f \in P(X) \colon Yf \subseteq Y\}$, since $P_{P(Y)}(X) = \{f \in P(X) \colon Yf \subseteq Y\}$. The semigroup $P_{P(Y)}(X)$ was recently introduced in both [2, p.2] and [13, p.124]. Moreover, both left and right magnifiers in $P_{P(Y)}(X)$ have been described (see, [2, Theorems 13, 20] and [13, Theorems 3.3, 4.4]).

The aim of this paper is to describe left and right magnifiers in the semigroup $P_{\mathbb{S}(Y)}(X)$, where Y is a nonempty subset of X and $\mathbb{S}(Y)$ is a general submonoid of $P(Y)$. The rest of the paper is organized as follows. In the next section, we define concepts, introduce notation, and state a necessary theorem. In Section 3, we describe left magnifiers in the semigroup $P_{\mathbb{S}(Y)}(X)$. We then apply this general result to describe left magnifiers in the semigroup $P_{\mathcal{I}(Y)}(X)$, where $\mathcal{I}(Y)$ is the symmetric inverse semigroup on Y. We also identify the corresponding known results on the semigroups $P_{P(Y)}(X)$ and $P_{\Omega(Y)}(X)$, where $\Omega(Y)$ is the subsemigroup of $P(Y)$ consisting of all partial surjective transformations on Y. In Section 4, we describe right magnifiers in $P_{\mathbb{S}(Y)}(X)$. We then apply this general result to describe right magnifiers in $P_{\mathcal{I}(Y)}(X)$. We also identify the corresponding known results $P_{P(Y)}(X)$ and $P_{\Omega(Y)}(X)$.

2. Preliminaries and notations

Let X be a nonempty set. For a nonempty subset Y of X, we write Y' to refer the set $X \setminus Y := \{x \in X \colon x \notin Y\}$. The identity transformation on X is denoted by id_X. A *partial transformation* on X is a mapping $f \colon A \to X$, where A is a subset of X. We write xf to denote the image of $x \in X$ under $f \in P(X)$. We compose elements of $P(X)$ from left to right and denote their composition by juxtaposition. For an element $f \in P(X)$, let $\operatorname{dom} f := \{x \in X \colon xf = y \text{ for some } y \in X\}$ and $\operatorname{im} f := \{xf \colon x \in X\}$.

Let $f \in P(X)$. For every $A \subseteq \operatorname{dom} f$ (resp. $B \subseteq \operatorname{im} f$), let Af (resp. Bf^{-1}) denote the set $\{af \colon a \in A\}$ (resp. $\{x \in X : xf \in B\}$). Moreover, we write bf^{-1} to denote the set $\{b\}f^{-1}$ if $B = \{b\}$. A subset Y of X is *invariant* under f if $Yf \subseteq Y$. If Y is invariant under f, then the *restriction* of f to Y is the self-map $f|_Y \colon Y \to Y$ defined by $x(f|_Y) = xf$.

Let X be a nonempty set. Denote by $\mathcal{I}(X)$ the symmetric inverse semigroup on X. The semigroup $\mathcal{I}(X)$ is fundamental in inverse semigroup theory, since every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [5, Theorem 5.1.7]. Let $\Omega(X)$ denote the semigroup under the composition consisting of all partial surjective transformations on X.

An element a of a semigroup S with identity e is a *left inverse* (resp. *right inverse*) of an element $b \in S$ if $ab = e$ (resp. $ba = e$). For any undefined notions and results on a semigroup, we refer the reader to [5]. In the rest of the paper, let Y be an infinite subset of a set X, and let $\mathbb{S}(Y)$ be a general submonoid of $P(Y)$. We end this section with the following theorem, which describes left and right magnifiers in any semigroup with identity.

Theorem 2.1. [10, Theorem 2.1] *Let S be a semigroup with identity e, and $a \in S$. Then:*

(i) *the element a is a left magnifier in S if and only if there exists $b \in S$ such that $ab = e$ but $ba \neq e$;*

(ii) *the element a is a right magnifier in S if and only if there exists $b \in S$ such that $ba = e$ but $ab \neq e$.*

3. Left magnifiers in $P_{\mathbb{S}(Y)}(X)$

In this section, we describe left magnifiers in the semigroup $P_{\mathbb{S}(Y)}(X)$. We apply this general result to describe left magnifiers in the semigroups $P_{\mathcal{I}(Y)}(X)$, $P_{\Omega(Y)}(X)$, and $P_{P(Y)}(X)$. We begin with a list of lemmas.

Lemma 3.1. *If f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$, then* $\operatorname{dom} f = X$ *and f is injective.*

Proof. Assume that f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$. Then, by Theorem 2.1(i) there exists $g \in P_{\mathbb{S}(Y)}(X)$ such that $fg = \mathrm{id}_X$. This implies that $\operatorname{dom} f = X$ and f is injective. □

Lemma 3.2. *Let $f \in P_{\mathbb{S}(Y)}(X)$. If $yf^{-1} \cap Y' \neq \varnothing$ for some $y \in Y \cap \operatorname{im} f$, then f is not a left magnifier in $P_{\mathbb{S}(Y)}(X)$.*

Proof. Assume that there exists $y \in Y \cap \operatorname{im} f$ such that $yf^{-1} \cap Y' \neq \varnothing$. Then $xf = y$ for some $x \in Y'$. Suppose to the contrary that f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$. Then, by Theorem 2.1(i) there exists $g \in P_{\mathbb{S}(Y)}(X)$ such that $fg = \mathrm{id}_X$. Since $y \in Y$ and $xf = y$, we obtain $x = x(fg) = (xf)g = yg$. This gives $x \in Y$, since $Yg \subseteq Y$. This is a contradiction to the fact that $x \in Y'$. □

Lemma 3.3. *Let $f \in P_{\mathbb{S}(Y)}(X)$. If $f|_Y$ has no right inverse in $\mathbb{S}(Y)$, then f is not a left magnifier in $P_{\mathbb{S}(Y)}(X)$.*

Proof. Assume that f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$. Then, by Theorem 2.1(i) there exists $g \in P_{\mathbb{S}(Y)}(X)$ such that $fg = \mathrm{id}_X$. Therefore $f|_Y g|_Y = \mathrm{id}_Y$. This shows that $g|_Y \in \mathbb{S}(Y)$ is a right inverse of $f|_Y$. □

Lemma 3.4. *Let $f \in P_{\mathbb{S}(Y)}(X)$ such that $\operatorname{dom} f = X$. If f is surjective, then f is not a left magnifier in $P_{\mathbb{S}(Y)}(X)$.*

Proof. Assume that f is surjective. Suppose to the contrary that f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$. Then there exists a proper subset M of $P_{\mathbb{S}(Y)}(X)$ such that $fM = P_{\mathbb{S}(Y)}(X)$. Also by Lemma 3.3, let $\gamma \in \mathbb{S}(Y)$ a right inverse of $f|_Y$. Note that f is injective by Lemma 3.1, and so f is bijective. Then by Lemma 3.2, the mapping $f|_Y$ is bijective. Thus $(f|_Y)^{-1} = \gamma$. Notice that $(f^{-1})|_Y = (f|_Y)^{-1}$ by Lemma 3.2, and so $(f^{-1})|_Y = \gamma$. This implies that $f^{-1} \in P_{\mathbb{S}(Y)}(X)$, since $\gamma \in \mathbb{S}(Y)$. Thus, since $fM = P_{\mathbb{S}(Y)}(X)$, we obtain $M = f^{-1}(fM) = f^{-1}P_{\mathbb{S}(Y)}(X) = P_{\mathbb{S}(Y)}(X)$. This is a contradiction to the fact that M is a proper subset of $P_{\mathbb{S}(Y)}(X)$. □

In the following theorem, we describe left magnifiers in the semigroup $P_{\mathbb{S}(Y)}(X)$.

Theorem 3.5. *Let $f \in P_{\mathbb{S}(Y)}(X)$. Then f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$ if and only if:*

(i) $\operatorname{dom} f = X$;

(ii) *f is injective but not surjective;*

(iii) $yf^{-1} \subseteq Y$ *for all* $y \in Y \cap \operatorname{im} f$;

(iv) $f|_Y$ *has a right inverse in* $\mathbb{S}(Y)$.

Proof. Assume that f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$. Then (i) holds by Lemma 3.1, while (ii) holds by Lemmas 3.1 and 3.4. Conditions (iii) and (iv) follow directly from Lemmas 3.2 and 3.3, respectively.

Conversely, assume that the given conditions hold. By (iv), let $h \in \mathbb{S}(Y)$ a right inverse of $f|_Y$. Recall from (ii) that f is injective, and define $g \in P(X)$ as follows:

$$yg = \begin{cases} yh & \text{if } y \in Y \cap \operatorname{im} f; \\ x & \text{if } y \in Y' \cap \operatorname{im} f, \text{ where } xf = y. \end{cases}$$

Notice that $\operatorname{dom} g = \operatorname{im} f$ and $g|_Y = h$. Since $h \in \mathbb{S}(Y)$, it follows that $g \in P_{\mathbb{S}(Y)}(X)$. Now we prove that $fg = \mathrm{id}_X$. Let $x \in X$.
Case 1: Suppose $x \in Y$. Then, since $xf = x(f|_Y)$ and $f|_Y h = \mathrm{id}_Y$, we obtain $x(fg) = (xf)g = (xf)h = (x(f|_Y))\, h = x(f|_Y h) = x$.
Case 2: Suppose $x \in Y'$. Then $x(fg) = x$.

In each case, we have $x(fg) = x$. Therefore $fg = \mathrm{id}_X$. Finally, since f is not surjective by (ii), we deduce that $gf \neq \mathrm{id}_X$. Thus f is a left magnifier in $P_{\mathbb{S}(Y)}(X)$ by Theorem 2.1(i). □

Corollary 3.6. *Let $f \in P_{\mathcal{I}(Y)}(X)$. Then f is a left magnifier in $P_{\mathcal{I}(Y)}(X)$ if and only if:*

(i) $\operatorname{dom} f = X$;

(ii) *Y' is invariant under f;*

(iii) *$f|_{Y'}$ is injective;*

(iv) *f is not surjective.*

Proof. Assume that f is a left magnifier in $P_{\mathcal{I}(Y)}(X)$. Then (i) holds by Theorem 3.5(i), while both (iii) and (iv) hold by Theorem 3.5(ii). Condition (ii) follows from Theorem 3.5(iii).

Conversely, assume that the given conditions hold. Note that $f|_Y$ is injective. Therefore f is injective by (ii) and (iii), and so $f^{-1} \in \mathcal{I}(X)$. Define $g \in P(X)$ by $g = f^{-1}$. By (ii), we have $yf^{-1} \subseteq Y$ for all $y \in Y$. Thus $g|_Y \in \mathcal{I}(Y)$, and so $g \in P_{\mathcal{I}(Y)}(X)$. It is clear that $fg = \mathrm{id}_X$, since $\operatorname{dom} f = X$ and $g = f^{-1}$. By (iv), we deduce that $gf \neq \mathrm{id}_X$. Thus f is a left magnifier in $P_{\mathcal{I}(Y)}(X)$ by Theorem 2.1(i). □

The next corollary follows directly from Theorem 3.5, which was proved in both [2, Theorem 20] and [13, Theorem 3.3].

Corollary 3.7. *Let $f \in P_{P(Y)}(X)$. Then f is a left magnifier in $P_{P(Y)}(X)$ if and only if $\operatorname{dom} f = X$, f is injective but not surjective, and $yf^{-1} \subseteq Y$ for all $y \in Y \cap \operatorname{im} f$.*

The next corollary was proved in [13, Theorem 5.3].

Corollary 3.8. *Let $f \in P_{\Omega(Y)}(X)$. Then f is a left magnifier in $P_{\Omega(Y)}(X)$ if and only if the following conditions hold:*

(i) $\operatorname{dom} f = X$;

(ii) *f is injective;*

(iii) $f|_{Y'}$ *is not surjective.*

Proof. Assume that f is a left magnifier in $P_{\Omega(Y)}(X)$. Then (i) holds by Theorem 3.5(i), while (ii) holds by Theorem 3.5(ii). Note that $f|_Y$ is surjective. Since f is not surjective by Theorem 3.5(ii), the condition (iii) follows.

Conversely, assume that the given conditions hold. Notice from (i) that $\operatorname{dom}(f|_Y) = Y$. By (ii), we then deduce that $Y'f \subseteq Y'$ and $f|_Y$ is bijective. Recall from (ii) that f is injective, and define $g \in P(X)$ by $xg = xf^{-1}$. Notice that $g|_Y = (f^{-1})|_Y = (f|_Y)^{-1}$. Since $(f|_Y)^{-1} \in \Omega(Y)$, it follows that $g|_Y \in \Omega(Y)$. Therefore $g \in P_{\Omega(Y)}(X)$. It is clear that $fg = \mathrm{id}_X$, since $\operatorname{dom} f = X$ and $g = f^{-1}$. Finally, by (iii) we deduce that $gf \neq \mathrm{id}_X$. Thus f is a left magnifier in $P_{\Omega(Y)}(X)$ by Theorem 2.1(i). □

4. Right magnifiers in $P_{\mathbb{S}(Y)}(X)$

In this section, we describe right magnifiers in the semigroup $P_{\mathbb{S}(Y)}(X)$. We apply this general result to describe right magnifiers in the semigroups $P_{\mathcal{I}(Y)}(X)$, $P_{\Omega(Y)}(X)$, and $P_{P(Y)}(X)$. We begin with a list of lemmas.

Lemma 4.1. *If f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$, then f is surjective.*

Proof. Assume that f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$. Then, by Theorem 2.1(ii) there exists $g \in P_{\mathbb{S}(Y)}(X)$ such that $gf = \mathrm{id}_X$. This implies that f is surjective. □

Lemma 4.2. *Let $f \in P_{\mathbb{S}(Y)}(X)$. If $yf^{-1} \cap Y = \varnothing$ for some $y \in Y$, then f is not a right magnifier in $P_{\mathbb{S}(Y)}(X)$.*

Proof. Assume that f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$. Then, by Theorem 2.1(ii) there exists $g \in P_{\mathbb{S}(Y)}(X)$ such that $gf = \mathrm{id}_X$. Let $y \in Y$. Then $yg \in Y$, since $g|_Y \in \mathbb{S}(Y)$. Thus $(yg)f = y(gf) = y$, and so $yg \in yf^{-1}$. Since $yg \in Y$, we conclude that $yf^{-1} \cap Y \neq \varnothing$. □

Lemma 4.3. *Let $f \in P_{\mathbb{S}(Y)}(X)$. If $f|_Y$ has no left inverse in $\mathbb{S}(Y)$, then f is not a right magnifier in $P_{\mathbb{S}(Y)}(X)$.*

Proof. Assume that f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$. Then, by Theorem 2.1(ii) there exists $g \in P_{\mathbb{S}(Y)}(X)$ such that $gf = \mathrm{id}_X$. This deduces that $g|_Y f|_Y = \mathrm{id}_Y$. Hence $g|_Y$ is a left inverse of $f|_Y$ in $\mathbb{S}(Y)$. □

Lemma 4.4. *Let f be a right magnifier in $P_{\mathbb{S}(Y)}(X)$. If $f|_Y$ has a left inverse in $\mathbb{S}(Y)$, then $yf^{-1} \cap Y \neq \varnothing$ for all $y \in Y$.*

Proof. Assume that $f|_Y$ has a left inverse in $\mathbb{S}(Y)$. Let $h \in \mathbb{S}(Y)$ a left inverse of $f|_Y$. Then $h(f|_Y) = \mathrm{id}_Y$. Let $y \in Y$. Then $yh \in Y$, since $h \in \mathbb{S}(Y)$. Thus $(yh)f = (yh)f|_Y = y(h(f|_Y)) = y$, and so $yh \in yf^{-1}$. Since $yh \in Y$, we conclude that $yf^{-1} \cap Y \neq \varnothing$. □

Lemma 4.5. *If f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$, then* $\operatorname{dom} f \neq X$ *or f is not injective.*

Proof. Assume that f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$. Then there exists a proper subset M of $P_{\mathbb{S}(Y)}(X)$ such that $Mf = P_{\mathbb{S}(Y)}(X)$. By Lemma 4.1, we also note that f is surjective. Suppose to the contrary that $\operatorname{dom} f = X$ and f is injective. Then f is bijective. Let $g \in P_{\mathbb{S}(Y)}(X)$. Clearly $gf \in P_{\mathbb{S}(Y)}(X)$. Since $Mf = P_{\mathbb{S}(Y)}(X)$, we deduce that $hf = gf$ for some $h \in M$. Thus, since $\operatorname{dom} f = X$ and f is bijective, we obtain $g = (gf)f^{-1} = (hf)f^{-1} = h$. Therefore $g \in M$, since $h \in M$. Thus, since $g \in P_{\mathbb{S}(Y)}(X)$, we conclude that $P_{\mathbb{S}(Y)}(X) \subseteq M$. This is a contradiction to the fact that M is a proper subset of $P_{\mathbb{S}(Y)}(X)$. □

Remark 4.6. Let $f \in P(X)$. If $\operatorname{dom} f \neq X$ or f is not injective, then $fg \neq \mathrm{id}_X$ for every $g \in P(X)$.

In the following theorem, we describe right magnifiers in the semigroup $P_{\mathbb{S}(Y)}(X)$.

Theorem 4.7. *Let $f \in P_{\mathbb{S}(Y)}(X)$. Then f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$ if and only if:*

(i) *f is surjective;*
(ii) *$f|_Y$ has a left inverse in $\mathbb{S}(Y)$;*
(iii) $\operatorname{dom} f \neq X$ *or f is not injective.*

Proof. Assume that f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$. Then (i), (ii), and (iii) follow directly from Lemmas 4.1, 4.3, and 4.5, respectively.

Conversely, assume that the given conditions hold. By (i), fix $z_x \in xf^{-1}$ for each $x \in X$. By (ii), let $\gamma \in \mathbb{S}(Y)$ be a left inverse of $f|_Y$. Define $g \in P(X)$ as follows:

$$xg = \begin{cases} x\gamma & \text{if } x \in Y; \\ z_x & \text{if } x \in Y'. \end{cases}$$

Notice that $g|_Y = \gamma$. Since $\gamma \in \mathbb{S}(Y)$, it follows that $g \in P_{\mathbb{S}(Y)}(X)$. Now by (iii) and Remark 4.6, we obtain $fg \neq \mathrm{id}_X$. Finally, we prove that $gf = \mathrm{id}_X$. Let $x \in X$.

CASE 1: Suppose $x \in Y$. Then $xg = x\gamma$. Thus, since $xf = x(f|_Y)$ and $\gamma(f|_Y) = \mathrm{id}_Y$, we obtain $x(gf) = (xg)f = (x\gamma)f = (x\gamma)f|_Y = x\left(\gamma(f|_Y)\right) = x$.

CASE 2: Suppose $x \in Y'$. Then $xg = z_x$. Thus, since $z_x \in xf^{-1}$, we obtain $x(gf) = (xg)f = z_x f = x$.

In each case, we have $x(gf) = x$. Therefore $gf = \mathrm{id}_X$. Thus f is a right magnifier in $P_{\mathbb{S}(Y)}(X)$ by Theorem 2.1(ii). □

Corollary 4.8. *Let $f \in P_{\mathcal{I}(Y)}(X)$. Then f is a right magnifier in $P_{\mathcal{I}(Y)}(X)$ if and only if:*

(i) *f is surjective;*

(ii) *$yf^{-1} \cap Y \neq \varnothing$ for all $y \in Y$;*

(iii) *$\operatorname{dom} f \neq X$ or $f|_{Y'}$ is not injective.*

Proof. Assume that f is a right magnifier in $P_{\mathcal{I}(Y)}(X)$. Then (i) holds by Theorem 4.7(i), while (ii) holds by Theorem 4.7(ii) and Lemma 4.4. Finally, since $f|_Y$ is injective, the condition (iii) follows from Theorem 4.7(iii).

Conversely, assume that the given conditions hold. By (i), choose $z_x \in xf^{-1}$ for each $x \in X$. By (ii), choose $y_x \in xf^{-1} \cap Y$ for each $x \in Y$. Define $g \in P(X)$ as follows:

$$xg = \begin{cases} y_x & \text{if } x \in Y; \\ z_x & \text{if } x \in Y'. \end{cases}$$

Note that $xf^{-1} \cap Y = \{y_x\}$ for every $x \in Y$, since $f|_Y$ is injective. This implies that $g|_Y \in \mathcal{I}(Y)$, and so $g \in P_{\mathcal{I}(Y)}(X)$. Now by (iii) and Remark 4.6, we obtain $fg \neq \mathrm{id}_X$. Finally, we prove that $gf = \mathrm{id}_X$. Let $x \in X$.

CASE 1: Suppose $x \in Y$. Then $xg = y_x$. Thus, since $y_x \in xf^{-1} \cap Y$, we obtain $x(gf) = (xg)f = y_x f = x$.

CASE 2: Suppose $x \in Y'$. Then $xg = z_x$. Thus, since $z_x \in xf^{-1}$, we obtain $x(gf) = (xg)f = z_x f = x$.

In each case, we have $x(gf) = x$. Therefore $gf = \mathrm{id}_X$. Thus f is a right magnifier in $P_{\mathcal{I}(Y)}(X)$ by Theorem 2.1 (ii). □

The following corollary was proved in both [2, Theorem 13] and [13, Theorem 4.4].

Corollary 4.9. *Let $f \in P_{P(Y)}(X)$. Then f is a right magnifier in $P_{P(Y)}(X)$ if and only if:*

(i) *f is surjective;*

(ii) *$yf^{-1} \cap Y \neq \varnothing$ for all $y \in Y$;*

(iii) *$\operatorname{dom} f \neq X$ or f is not injective.*

Proof. Assume that f is a right magnifier in $P_{P(Y)}(X)$. Then (i), (ii), and (iii) follow directly from Lemmas 4.1, 4.2, and 4.5, respectively.

Conversely, assume that the given conditions hold. By (i), fix $z_x \in xf^{-1}$ for each $x \in X$. By (ii), fix $y_x \in xf^{-1} \cap Y$ for each $x \in Y$. Define $g \in P(X)$ as follows:

$$xg = \begin{cases} y_x & \text{if } x \in Y; \\ z_x & \text{if } x \in Y'. \end{cases}$$

Notice that $g|_Y \in P(Y)$, since $y_x \in Y$ and $x(g|_Y) = y_x$ for every $x \in Y$. Therefore $g \in P_{P(Y)}(X)$. Now by (iii) and Remark 4.6, we obtain $fg \neq \mathrm{id}_X$. Finally, we prove that $gf = \mathrm{id}_X$. Let $x \in X$.
Case 1: Suppose $x \in Y$. Then $xg = y_x$. Thus, since $y_x \in xf^{-1}$, we obtain $x(gf) = (xg)f = y_x f = x$.
Case 2: Suppose $x \in Y'$. Then $xg = z_x$. Thus, since $z_x \in xf^{-1}$, we obtain $x(gf) = (xg)f = z_x f = x$.

In each case, we have $x(gf) = x$. Therefore $gf = \mathrm{id}_X$. Thus f is a right magnifier in $P_{P(Y)}(X)$ by Theorem 2.1(ii). □

The next two corollaries was discussed in [13, Theorem 6.3].

Corollary 4.10. *Let $f \in P_{\Omega(Y)}(X)$ such that* $\operatorname{dom} f = X$. *Then f is a right magnifier in $P_{\Omega(Y)}(X)$ if and only if:*

(i) *$f|_{Y'}$ is surjective;*

(ii) *f is not injective.*

Proof. Assume that f is a right magnifier in $P_{\Omega(Y)}(X)$. Then f is surjective by Theorem 4.7(i), and so $f|_{Y'}$ is surjective. Condition (ii) follows from Lemma 4.5, since $\operatorname{dom} f = X$.

Conversely, assume that the given conditions hold. Then f is surjective by (i), since $f|_Y \in \Omega(Y)$. Choose $z_x \in xf^{-1}$ for each $x \in X$. Notice that

$xf^{-1} \cap Y \neq \varnothing$ for each $x \in Y$, since $f|_Y \in \Omega(Y)$. Fix $y_x \in xf^{-1} \cap Y$ for each $x \in Y$, and define $g \in P(X)$ as follows:

$$xg = \begin{cases} y_x & \text{if } x \in Y; \\ z_x & \text{if } x \in Y'. \end{cases}$$

Notice that $\operatorname{dom}(f|_Y) = Y$, since $\operatorname{dom} f = X$. Thus $\operatorname{im}(g|_Y) = \operatorname{dom}(f|_Y) = Y$, and so $g|_Y \in \Omega(Y)$. Therefore $g \in P_{\Omega(Y)}(X)$. Now we prove that $gf = \mathrm{id}_X$. Let $x \in X$.
Case 1: Suppose $x \in Y$. Then $xg = y_x$. Thus, since $y_x f = x$, we obtain $x(gf) = (xg)f = y_x f = x$.
Case 2: Suppose $x \in Y'$. Then $xg = z_x$. Thus, since $z_x f = x$, we obtain $x(gf) = (xg)f = z_x f = x$.

In each case, we have $x(gf) = x$. Therefore $gf = \mathrm{id}_X$. By (ii) and Remark 4.6, we deduce that $fg \neq \mathrm{id}_X$. Thus f is a right magnifier in $P_{\Omega(Y)}(X)$ by Theorem 2.1(ii). □

Corollary 4.11. *Let $f \in P_{\Omega(Y)}(X)$ such that $\operatorname{dom} f \neq X$. Then f is a right magnifier in $P_{\Omega(Y)}(X)$ if and only if:*

(i) *$f|_{Y'}$ is surjective;*

(ii) *$f|_Y$ has a left inverse in $\Omega(Y)$.*

Proof. This follows directly from Theorem 4.7. □

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