

Recognition of the simple group $Sz(q)$, for some q by character degree graph and order

Zahra Momen

Abstract. Let G be a finite group. The character degree graph of G , which is denoted by $\Gamma(G)$, is the graph whose vertices are the prime divisors of the irreducible character degrees of G and two vertices p and p' are joined by an edge if pp' divides some irreducible character degree of G . In this paper, we prove that the simple group $Sz(q)$, for some q , is uniquely determined by its character degree graph and its order. This implies that $Sz(q)$ is uniquely determined by the structure of its complex group algebra.

1. Introduction

Let G be a finite group, $\text{Irr}(G)$ be the set of irreducible characters of G , and denote by $\text{cd}(G)$, the set of irreducible character degrees of G . The *character degree graph* of G , which is denoted by $\Gamma(G)$, is defined as follows: the vertices of this graph are the prime divisors of the irreducible character degrees of G and two distinct vertices p and p' are joined by an edge if there exists an irreducible character degree of G which is divisible by pp' . This graph has been widely studied (see [17, 19]). A finite group G is called a K_3 -group if $|G|$ has exactly three distinct prime divisors. Recently Chen et al. in [20] and [21] proved that all simple K_3 -groups and the Mathieu groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees.

Let p be an odd prime number. In [9], Khosravi et al. proved that the simple group $\text{PSL}(2, p)$ is uniquely determined by its order and its largest and second largest irreducible character degrees. Also as a consequence of their result it follows that the simple group $\text{PSL}(2, p)$ is uniquely determined by its character degree graph and its order (also see [12, 18]). In [1], it is proved that the simple group $\text{PSL}(2, q)$ is recognized by character degree

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graph and order. Also in [8], it is proved that the characteristically simple group $A_5 \times A_5$ is recognizable by character degree graph and order. Also in [13] it is proved that each simple group of order less than 6000 is uniquely determined by its character degree graph and its order. In [14] it is proved that if p is an odd prime number and G is a finite group such that $|G| = |\text{PSL}(2, p^2)|$, $p^2 \in \text{cd}(G)$ and there does not exist any $\theta \in \text{Irr}(G)$ such that $2p \mid \theta(1)$, then $G \cong \text{PSL}(2, p^2)$. In [10], Khosravi et al. proved that if G is a finite group such that $|G| = 2|\text{PSL}(2, p^2)|$, $p^2 \in \text{cd}(G)$ and there does not exist any $\theta \in \text{Irr}(G)$ such that $2p \mid \theta(1)$, then G has a unique nonabelian composition factor isomorphic to $\text{PSL}(2, p^2)$. Similar results were proved for some extensions of $\text{PSL}(2, q)$, for some q (see [11]). In [7] it is proved that the projective special linear group $\text{PSL}(2, q)$ is uniquely determined by its group order and its largest irreducible character degree when q is a prime or when $q = 2^a$ for an integer $a \geq 2$ such that $2^a - 1$ or $2^a + 1$ is a prime.

In [15] it is proved that $\text{PSL}(2, p^2)$, where p is a prime, is uniquely determined by its order and its character degree graph. In [2], it is proved that some characteristically simple groups are recognizable by their complex group algebras. In [16], we proved that $\text{PSL}(2, p) \times \text{PSL}(2, p)$ is uniquely determined by the structure of its complex group algebra. In [4], it is proved that Suzuki groups $\text{Sz}(q)$, where $q \pm \sqrt{2q} + 1$ is a prime number can be uniquely determined by the order of group and the number of elements with the same order. In this paper, we prove that the simple group $\text{Sz}(q)$, where $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ is a prime number is uniquely determined by its character degree graph and its order.

2. Preliminary results

Definition 2.1. If $N \trianglelefteq G$ and $\theta \in \text{Irr}(N)$, then the inertia group of θ in G is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^k e_i \chi_i$, where for each $1 \leq i \leq k$, $\chi_i \in \text{Irr}(G)$ and e_i is a natural number, then each χ_i is called an irreducible constituent of χ .

Lemma 2.2. (Itô's Theorem) [6, Theorem 6.15] *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$, for all $\chi \in \text{Irr}(G)$.*

Lemma 2.3. [6, Theorems 6.2, 6.8, 11.29] *Let $N \trianglelefteq G$ and let $\chi \in \text{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose $\theta_1 = \theta, \dots, \theta_t$ are the distinct conjugates of θ in G . Then $\chi_N = e \sum_{i=1}^t \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 2.4. [20, Lemma] *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\text{Out}(K/H)|$.*

Lemma 2.5. (Itô-Michler Theorem) [5] *Let $\rho(G)$ be the set of all prime divisors of the elements of $\text{cd}(G)$. Then $p \notin \rho(G)$ if and only if G has a normal abelian Sylow p -subgroup.*

Lemma 2.6. [21, Lemma 2] *Let G be a finite solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1, p_2, \dots, p_n$ are distinct primes. If $(kp_n + 1) \nmid p_i^{\alpha_i}$, for each $i \leq n - 1$ and $k > 0$, then the Sylow p_n -subgroup is normal in G .*

If n is an integer and r is a prime number, then we write $r^\alpha \parallel n$, when $r^\alpha \mid n$ but $r^{\alpha+1} \nmid n$. Also if r is a prime number we denote by $\text{Syl}_r(G)$, the set of Sylow r -subgroups of G and we denote by $n_r(G)$, the number of elements of $\text{Syl}_r(G)$. If H is a characteristic subgroup of G , we write $H \text{ ch } G$. All other notations are standard and we refer to [3].

3. The main results

Remark 3.1. [17] Let $G = \text{Sz}(q)$, where $q = 2^{2m+1} \geq 8$. Throughout of this section let $a = q + \sqrt{2q} + 1$ and $b = q - \sqrt{2q} + 1$. Then $|G| = q^2(q^2 + 1)(q - 1)$ and $\text{cd}(G) = \{1, q^2, q^2 + 1, (q - 1)a, (q - 1)b, 2^m(q - 1)\}$. It is obvious that $(a, b) = 1$, $(q^2 + 1, q - 1) = 1$ and $(q^2 + 1, q) = 1$.

Lemma 3.2. *Let $q = 2^{2m+1}$ and $b = q - \sqrt{2q} + 1$, where $m > 1$ is an integer. Then*

1. $(kb + 1) \nmid q^2$, for every $k \in \mathbb{N}$.
2. *There is not any divisor d of $a = q + \sqrt{2q} + 1$, where $d = kb + 1$, for some $k \in \mathbb{N}$.*
3. *There is not any divisor d of $q - 1$, where $d = kb + 1$, for some $k \in \mathbb{N}$.*

Proof. 1. Since $b - 1 \equiv 2^{m+1} \pmod{2^m - 1}$, so $2, 2^2, \dots, 2^{2m} \in [2, b - 1]$. Therefore $\text{ord}_b 2 \notin [2, b - 1]$. It is obvious that

$$2^{2m+1} \equiv 2^{m+1} - 1 \pmod{b} \quad (1)$$

Also $2^{m+1} - 1 \leq b - 1 = 2^{m+1}(2^m - 1)$. So $2^{2m+1} \in [2, b - 1]$. Now by (1), we have $2^{3m} \equiv 2^{2m} - 2^{m-1} \pmod{b}$. Hence $3m < \text{ord}_b 2$. Continuing in this manner we conclude that $2^{3m+1}, 2^{3m+2}, \dots, 2^{4m+1} \in [2, b - 1]$.

Now we claim that $2^{4m+2} \equiv -1 \pmod{b}$. Using (1), we have $2^{4m+1} \equiv 2^{3m+1} - 2^{2m} \pmod{b}$. Therefore $2^{4m+2} \equiv 2^{2m+2} - 2^{m+1} - 2^{2m+1} \equiv -1 \pmod{b}$. Up to now, we prove that $\text{ord}_b 2 > 4m + 2$. On the other hand, $b \mid (q^4 - 1)$. Hence $\text{ord}_b 2 \mid 4(2m + 1)$. Thus $\text{ord}_b 2 = 8m + 4$.

If $(kb + 1) \mid q^2$, for some $k \geq 1$, then $kb + 1 = 2^t$ for some t such that $t \leq 4m + 2$. So $2^t \equiv 1 \pmod{b}$, which is a contradiction. Hence the proof is completed.

2. We know that $q + 1 > 3 \cdot 2^{m+1} = 3\sqrt{2q}$. Thus $2b > a$ and so $(kb + 1) \nmid a$, for any $k \geq 1$.

3. Since $2b > (q - 1)$, so $q + 3 > 2\sqrt{2q}$. Therefore $kb + 1 > q - 1$ and so $(kb + 1) \nmid (q - 1)$. $\square$

Lemma 3.3. *Let $q = 2^{2m+1}$ and $a = q + \sqrt{2q} + 1$, where $m \geq 1$ is an integer. Then $(ka + 1) \nmid q^2$, for every $k \in \mathbb{N}$.*

Proof. Similarly to the proof of Lemma 3.2, we can get the result. $\square$

Lemma 3.4. *Let $q' = 2^{2\alpha+1}$ and $q = 2^{2\beta+1}$. Then $|\text{Sz}(q')| \mid |\text{Sz}(q)|$ if and only if $q = q'^s$, for some odd s .*

Proof. First suppose that $|\text{Sz}(q')| \mid |\text{Sz}(q)|$. Let p_1 be a primitive prime divisor of $2^{(2\alpha+1)} - 1$, i.e. $\text{ord}_{p_1} 2 = 2\alpha + 1$. Therefore by Remark 3.1, $p_1 \mid (q - 1)$ or $p_1 \mid (q^2 + 1)$. If $p_1 \mid (q - 1)$, then $(2\alpha + 1) \mid (2\beta + 1)$. Now suppose that $p_1 \mid (q^2 + 1)$. So $p_1 \mid (2^{4(2\beta+1)} - 1)$. Therefore $(2\alpha + 1) \mid (2\beta + 1)$. Consequently, $q = q'^s$, for some odd s .

If $q = q'^s$, for some odd s , then by Remark 3.1, the proof is obvious. $\square$

Theorem 3.5. *Let $q = 2^{2m+1}$ and $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ be a prime number, where $m \in \mathbb{N}$. Also let G be a finite group of order $|\text{Sz}(q)|$ such that $\Gamma(G) = \Gamma(\text{Sz}(q))$, then $G \cong \text{Sz}(q)$.*

Proof. Step 1. Let $q - \sqrt{2q} + 1 = p$ or $q + \sqrt{2q} + 1 = p$ be a prime number. Then we prove that $O_p(G) = 1$. For convenience, we give the proof only when $q + \sqrt{2q} + 1 = p$ be a prime number. Let $O_p(G) \neq 1$. By Remark 3.1, $|O_p(G)| = p$. Using Lemma 2.2, $p \mid \chi(1) \mid [G : O_p(G)]$, for all $\chi \in \text{Irr}(G)$, which is a contradiction. Therefore $O_p(G) = 1$.

Step 2. We prove that G is a nonsolvable group. Using Lemmas 2.6, 3.2 and 3.3, the Sylow p -subgroup is normal in G , which is a contradiction by Step 1. Therefore G is a nonsolvable group.

Step 3. We prove that $G \cong \text{Sz}(q)$. By Step 2 and Lemma 2.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\text{Out}(K/H)|$. Since $3 \nmid |\text{Sz}(2^{2m+1})|$, for $m \geq 1$, so $K/H \cong \text{Sz}(q') \times \text{Sz}(q') \times \dots \times \text{Sz}(q')$, the direct product of n -copies of $\text{Sz}(q')$, such that $q' \mid q$. If $n > 1$, then since $p \mid |G|$, so $p \nmid |K/H|$. Therefore $p \mid |G/K|$ or $p \mid |H|$.

Now let $p \mid |G/K|$. Note that $\text{Out}(K/H) \cong \text{Out}(\text{Sz}(q')) \wr S_n$. Since $|\text{Out}(K/H)| \mid (2m+1)^n \times n!$ and $p > 2m+1$ is prime, so $p \nmid (2m+1)^n$. Hence $p \mid |S_n|$, i.e. $p < n$. We know that $29120 = |\text{Sz}(8)| \leq |\text{Sz}(q')|$. Consequently, $29120^p \leq |K/H| < |G|$, which is a contradiction since $p \mid |G|$.

Therefore $p \mid |H|$. Since $H \trianglelefteq G$, so $p \mid \theta(1)$, for some irreducible character θ of H by Lemma 2.3. Now we consider the following cases:

(a) Let H be solvable. We denote by T , the p -Sylow subgroup of H . Using Lemmas 2.6, 3.2 and 3.3, $T \trianglelefteq G$, which is a contradiction.

(b) Let H be nonsolvable. By Lemma 2.4, G has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that B/A is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\text{Out}(B/A)|$. Similarly to the above, we can prove that $p \nmid |H/B|$ and $p \nmid |B/A|$. So similarly $p \mid |A|$ and A is solvable, which is a contradiction by Lemmas 2.6, 3.2 and 3.3.

Consequently, $p \mid |K/H|$ and $n = 1$. Thus $K/H \cong \text{Sz}(q')$, where $q' \mid q$, by Lemma 3.4. Now since $p \mid |G|$ and $p \mid |K/H|$, so $|G| = |K/H|$. Therefore $G \cong \text{Sz}(q)$. $\square$

As a consequence we get the following result:

Corollary 3.6. *Let $q = 2^{2m+1} \geq 8$ and $H = \text{Sz}(q)$. Also let $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ be a prime number. If G is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong \text{Sz}(q)$. Thus $\text{Sz}(q)$ is uniquely determined by the structure of its complex group algebra.*

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Department of Mathematics Education
Farhangian University
P.O. Box 14665-889
Tehran, Iran
E-mail: z.momen@cfu.ac.ir