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On the commutativity of prime MA-semirings using generalized reverse derivations

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Abstract. Motivated by the work done by Ashraf et al. [5] and Quadri et al. [17] this paper aims to investigate some significant features of generalised reverse derivations that ensure the commutativity of a prime MA-semiring.

1. Introduction

Let $\mathfrak{R}$ be a ring and $Z(\mathfrak{R})$ be the center of $\mathfrak{R}$. A mapping $\delta : \mathfrak{R} \to \mathfrak{R}$ is called a derivation if it satisfies the following properties: (i) $\delta(x+y) = \delta(x)+\delta(y)$; (ii) $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathfrak{R}$. Posner [16] proposed the concept of derivation and set up the connection between the commutativity of a ring and derivation. In the same year, I. N. Herstein [9] suggested the idea of reverse derivation. Since then, authors have introduced various types of derivations such as Jordan derivation, generalized derivation, and generalized reverse derivation and studied them (see [1], [7], [9], [11], [12], [18], [20] where further references can be found).

Ashraf and Rehmann [4] proved that if a derivation δ on a prime ring $\mathfrak{R}$ satisfies either of the properties $\delta(xy) + xy \in Z(\mathfrak{R})$ or $\delta(xy) - xy \in Z(\mathfrak{R})$ for all $x, y \in \mathcal{I}$ where $\mathcal{I}$ is a non-zero ideal of $\mathfrak{R}$, then $\mathfrak{R}$ must be commutative. After that, Ashraf et al. [5] extended this concept through generalized derivation. Ashraf et al. [5] and Quadri et al. [17] proved that if a generalized derivation F satisfies any of the following properties:

(i) $F(xy) + xy \in Z(\mathfrak{R})$,
(ii) $F(xy) - xy \in Z(\mathfrak{R})$,
(iii) $F(xy) + yx \in Z(\mathfrak{R})$,
(iv) $F(xy) - yx \in Z(\mathfrak{R})$,

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(v) $F(x)F(y) + xy \in Z(\mathfrak{R})$,
(vi) $F(x)F(y) - xy \in Z(\mathfrak{R})$,
(vii) $F(x)F(y) + yx \in Z(\mathfrak{R})$,
(viii) $F(x)F(y) - yx \in Z(\mathfrak{R})$,
(ix) $F(xy) = [x, y],\ (x)F(xy) = x \circ y$

for all $x, y \in \mathcal{I}$ where $\mathcal{I}$ is a non-zero ideal of $\mathfrak{R}$, then $\mathfrak{R}$ must be commutative.

As the theory of semirings not only generalises the theory of rings, but also has significant applications in optimization theory, graph theory, formal language theoretical computer science, and other areas of applied mathematics (see [8]), over the past ten years, many authors have extended the study of derivation in rings to the setting of semirings by exploring the interplay between the commutativity of a semiring $\mathcal{S}$ and specific types of derivations in $\mathcal{S}$. Here it is noteworthy that the commutativity of a semiring plays an important role in the field of research, e.g. the semiring $(R_{max}, \oplus, \otimes)$, where $R_{max} = \mathbb{R} \cup \{-\infty\}$, $a \oplus b = max\{a, b\}$ $a \otimes b = a + b$, proves to be very effective when applied to the idempotent analysis and becomes a staple tool in hundreds of optimization publications [15]. This makes the study of commutativity of semirings interesting and relevant as well. Recently, Yaqoub et al. [3] extended the notion of generalized reverse derivation as an additive mapping F from a semiring $\mathcal{S}$ to itself satisfying $F(xy) = F(y)x + y\delta(x)$ for all $x, y \in \mathcal{S}$, where δ is a reverse derivation of $\mathcal{S}$ and demonstrates how the features of this generalized reverse derivation influenced the commutativity of prime and semiprime additively inverse semirings.

Taking impetus from the work done by Ashraf et al. in 2007, and Quadri et al. in 2003, the aim of this paper is to investigate some significant features of generalized reverse derivations (other than those studied by Yaqoub [3]) which compel the commutativity of a prime MA-semiring.

2. Preliminaries

A *semiring* is defined as a non-empty set $\mathcal{S}$ equipped with two binary operations, addition and multiplication, such that both the additive and multiplicative reducts form semigroups, and multiplication distributes over addition from either side [14]. $(\mathcal{S}, +, .)$ is said to be a *semiring with zero* if it has an absorbing zero, i.e. $a0 = 0 = 0a$ and $a + 0 = 0 + a = 0$ for all $a \in \mathcal{S}$ and $\mathcal{S}$ is said to be commutative if multiplicative reduct $(\mathcal{S}, .)$ is commutative.

For each element x in a semiring $\mathcal{S}$, if there exists an element x' such that $x = x + x' + x$ and $x' = x' + x + x'$ then $\mathcal{S}$ is called an *additively regular semiring.* If x' is unique then $\mathcal{S}$ is called an *additively inverse semiring.* According ([14], [2]), for each x, y belong to an additively inverse semiring $\mathcal{S}$ we have $x = (x')'$, $(x + y)' = y' + x'$, $(xy)' = x'y = xy'$, $x.y = x'.y'$ and the following identity holds for all $a, b \in \mathcal{S}$:

$$a + b = 0 \ \ implies \ \ a = b'. \tag{1}$$

H. J. Bandelt and M. Petrich [6] showed that a semiring $\mathcal{S}$ whose additive reduct $(\mathcal{S}, +)$ is a regular semigroup can be expressed as a subdirect product of distributive lattice and a ring if and only if $(\mathcal{S}, +)$ is commutative and the following conditions hold: (A_1) $y(x + x') = x' + x$ for all $x, y \in \mathcal{S}$; (A_2) $y(x + x') = (x + x')y$ for all $x, y \in \mathcal{S}$; (A_3) $x + (x + x')y = x$ for all $x, y \in \mathcal{S}$; (A_4) if $x \in \mathcal{S}$ and $y + x = y$ for some $y \in \mathcal{S}$, then $x + x = x$. If an additively inverse semiring $\mathcal{S}$ which is additively commutative and contains zero is said to be an *MA-semiring* (cf. [13]) if it satisfies the condition (A_2) mentioned above, i.e. $a + a' \in Z(\mathcal{S})$ for all $a \in \mathcal{S}$. A subsemiring / ideal $\mathcal{T}$ of an MA-semiring $\mathcal{S}_1$ is called an *MA-subsemiring / MA-ideal* if for each $a \in \mathcal{T}$, $a' \in \mathcal{T}$ (cf. [13]). An MA-semiring $\mathcal{S}$ is a *prime MA-semiring* if $a\mathcal{S}b = 0$ then implies that either $a = 0$ or $b = 0$. An MA-semiring $\mathcal{S}$ is called a 2*-torsion free MA-semiring* if $2a = 0$ implies that $a = 0$ where $a \in \mathcal{S}$.

The *commutator* of elements x, y in an MA-semiring $\mathcal{S}$ is defined as $[x, y] = xy + y'x$, and the *anti commutator* is defined as $x \circ y = xy + yx$. In an MA-semiring $\mathcal{S}$, $[x, y] = 0$ implies that $xy = yx$ for all $x, y \in \mathcal{S}$.
We now recall the following results from [13], [2] and [19] for their use in the sequel.

Known facts 2.1

- ([13], Theorem 3.2) *If $\mathcal{S}$ is an MA-semiring, then for all $x, y, z \in \mathcal{S}$, the following identities are valid:*

  (i) $[x, y]' = [x, y'] = [x', y] = [y, x]$,

  (ii) $[x', y'] = [x, y] \quad [x, yx] = [x, y]x, \quad [xy, x] = x[y, x]$.

  (iii) $[xy, z] = x[y, z] + [x, z]y$

  (iv) $[x, yz] = y[x, z] + [x, y]z$.

- ([2], Lemma 3.1) *If $\delta : \mathcal{S} \to \mathcal{S}$ is an additive mapping on an additively inverse semiring $\mathcal{S}$ then $\delta(x') = \delta(x)'$ for all $x \in \mathcal{S}$.*

- ([19], Lemma 4) *Let $\mathcal{S}$ be a 2-torsion-free prime MA-semiring. If $a \in \mathcal{S}$ such that $[a,[a,s]] = 0$ for all $s \in \mathcal{S}$, then $[a,\mathcal{S}] = 0$.*

- ([3], Lemma 2.1) *In an MA-semiring $\mathcal{S}$, $[x,y] = 0$ implies that $xy = yx$ for all $x, y \in \mathcal{S}$.*

We refer the readers to [3] for the definitions of a derivation and a reverse derivation on a semiring.

3. Main results

In this section, we study how some significant features of generalized reverse derivations influence the commutativity of a prime MA-semiring. To accomplish this, we prove the following Lemmas.

Lemma 3.1. *Let $\mathcal{S}$ be an MA-semiring. If δ is a non-zero reverse derivation then $\delta(Z'(\mathcal{S})) \subset Z'(\mathcal{S}) \subset Z(\mathcal{S})$ where $Z'(\mathcal{S}) = \{a \in \mathcal{S} : [a,b] = 0$ for all $b \in \mathcal{S}\}$.*

Proof. Let $a \in Z'(\mathcal{S})$. Then $[a,s] = 0$ for all $s \in \mathcal{S}$. Thus, $\delta([a,s])=0$ for all $s \in \mathcal{S}$. We have $\delta(as + s'a) = \delta(s)a + s\delta(a) + \delta(a)s' + a\delta(s') = 0$ for all $s \in \mathcal{S}$. Using result 2.1, we get $\delta(s)a + s\delta(a) + \delta(a)s' + a(\delta(s))' = 0$ for all $s \in \mathcal{S}$. This implies $[\delta(s),a] + [s,\delta(a)] = 0$ for all $s \in \mathcal{S}$. Hence, $[s,\delta(a)] = 0$ for all $s \in \mathcal{S}$. Therefore $\delta(a) \in Z'(\mathcal{S})$ and hence $\delta(Z'(\mathcal{S}))$ is contained in the center $Z(\mathcal{S})$. □

Observation 3.2. Before going further, we mention here that in any ring $\mathcal{R}$, existence of an element a in $Z(\mathcal{R})$ implies $a \in Z'(\mathcal{R})$ where $Z'(\mathcal{R}) = \{a \in \mathcal{R} : [a,b] = 0$ for all $b \in \mathcal{R}\}$. But this may not hold in a 2-torsion free prime MA-semiring, even if the semiring is commutative, which is evident from the following example. This makes the study of derivation in the setting of MA-semiring far more difficult.

Example 3.3. Let $\mathcal{S} = (R_{max}, \oplus, \otimes)$ where $R_{max} = \mathbb{R} \cup \{-\infty\}$, and operation in given by $a \oplus b = max\{a,b\}$, $a \otimes b = a + b$. Then $\mathcal{S}$ is a 2-torsion free commutative prime MA-semiring in which $\mathcal{S} = Z(\mathcal{S})$ but $Z'(\mathcal{S}) = \{0\}$.

Lemma 3.4. *If $[a,b] = 0$ for all $a, b \in \mathcal{I}$, where $\mathcal{I}$ is a non-zero MA-ideal of a prime MA-semiring $\mathcal{S}$, then $\mathcal{S}$ is commutative.*

Proof. Substituting bs in place of b, we get $b[a,s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. This shows that $\mathcal{I}[a,s] = 0$ for all $a \in \mathcal{I}$, for all $s \in \mathcal{S}$. Since $\mathcal{S}$ is prime and $\mathcal{I} \neq 0$, the last implies $[a,s] = 0$ for all $a \in \mathcal{I}$, for all $s \in \mathcal{S}$. Again replacing a by au, $a[u,s] + [a,s]u = 0$ and using the last equation, we get $a[u,s] = 0$ for all $a \in \mathcal{I}$, for all $u, s \in \mathcal{S}$, i.e. $\mathcal{I}[u,s] = 0$. Again as $\mathcal{I}$ is non-zero and $\mathcal{S}$ is prime, we obtain $[u,s] = 0$ for all $u, s \in \mathcal{S}$. Hence, $\mathcal{S}$ is commutative. □

We immediately deduce the following Corollary.

Corollary 3.5. *If $b[a,b] = 0$ for all $a, b \in \mathcal{I}$, where $\mathcal{I}$ is a non-zero MA-ideal of a prime MA-semiring $\mathcal{S}$, then $\mathcal{S}$ is commutative.*

Corollary 3.6. *If $a[a,b] = 0$ for all $a, b \in \mathcal{I}$, where $\mathcal{I}$ is a non-zero MA-ideal of a prime MA-semiring $\mathcal{S}$, then $\mathcal{S}$ is commutative.*

Lemma 3.7. *Let $\mathcal{S}$ be a 2-torsion free prime MA-semiring and $\mathcal{I}$ be a non-zero MA-ideal of $\mathcal{S}$. If $[a,[a,b]] = 0$ for all $a, b \in \mathcal{S}$, then $\mathcal{I}$ is commutative.*

Proof. It is easy to observe that fact 2.1 holds for a non-zero MA-ideal of $\mathcal{S}$, too, i.e. if $a \in \mathcal{I}$ such that $[a,[a,b]] = 0$ for all $b \in \mathcal{I}$, then $[a, \mathcal{I}] = 0$. Hence, $[a,b] = 0$ for all $a, b \in \mathcal{I}$. Using Lemma 3.4, we get $\mathcal{S}$ is commutative. □

Now we try to extend Lemma 3 of [16] in the setting of a 2-torsion free prime MA-semiring considering reverse derivation instead of derivation.

Theorem 3.8. *For each $a \in \mathcal{S}$, if $[a, \delta(a)] = 0$, where δ is a non-zero reverse derivation of a 2-torsion free prime MA-semiring $\mathcal{S}$ which is not a ring, then $\mathcal{S}$ is commutative.*

Proof. By linearization of $[a, \delta(a)] = 0$, we get $[a, \delta(b)] + [b, \delta(a)] = 0$ for all $a, b \in \mathcal{S}$. Substituting b by ag and ga, we respectively obtain the following two equations

$$[a, \delta(g)]a + [a,g]\delta(a) + a[g, \delta(a)] = 0 \tag{2}$$

$$\delta(a)[a,g] + a[a, \delta(g)] + [g, \delta(a)]a = 0 \tag{3}$$

for all $a, g \in \mathcal{S}$. First, we take inverse both side of Equation (3) then adding equations (2), (3), we get

$$[[a,g], \delta(a)] + [[a, \delta(g)], a] + [a, [g, \delta(a)]] = 0 \tag{4}$$

Again replacing a by $a + [a,g]$ in the given hypothesis, we get

$$[[a,g], \delta(a)] + [a, [\delta(g), a]] + [a, [g, \delta(a)]] = 0 \tag{5}$$

for all $a, g \in \mathcal{S}$. Combining the last two equations, we get $[[a, \delta(g)], a] + [a, [\delta(g), a]]' = 0$ for all $a, g \in \mathcal{S}$. Thus, we have $[[a, \delta(g)], a] + [a, [\delta(g), a]]' = [[a, \delta(g)], a] + [[\delta(g), a], a] = [a, \delta(g)]a + a'[a, \delta(g)] + [\delta(g), a]a + a'[\delta(g), a] = a\delta(g)a + \delta(g)a'a + a'a\delta(g) + a'\delta(g)a' + \delta(g)aa + a'\delta(g)a + a'\delta(g)a + a'a'\delta(g) = 0$ for all $a, g \in \mathcal{S}$. Since $(a + a') \in Z(\mathcal{S})$, we obtain $\delta(g)[a, a] + [a, a]\delta(g) + \delta(g)[a, a] + \delta(g)[a, a] = 0$ for all $a, g \in \mathcal{S}$. Since $[a, a] \in Z(\mathcal{S})$, $4\delta(g)[a, a] = 0$ for all $a, g \in \mathcal{S}$. Since $\mathcal{S}$ is 2-torsion free, $\delta(g)[a, a] = 0$ for all $a, g \in \mathcal{S}$. Thus, $[\delta(g)[a, a], y] = 0$ for all $a, g, y \in \mathcal{S}$. Suppose $[a, a] = 0$ for all $a \in \mathcal{S}$ i.e. $a\mathcal{S}(a + a') = 0$. As $\mathcal{S}$ is prime, for each non-zero a in $\mathcal{S}$, $a + a' = 0$. Hence from the definition MA-semiring we conclude that $\mathcal{S}$ is a ring. Thus we arrive at a contradiction. So, there exists a non-zero a in $\mathcal{S}$ for which $[a, a] \neq 0$. Since $\mathcal{S}$ is prime MA-semiring and $[a, a]$ is non-zero,

$$[\delta(g), y] = 0 \tag{6}$$

for all $y, g \in \mathcal{S}$. Again replacing g by vy, we get $[\delta(y)v + y\delta(v), y] = \delta(y)[v, y] + [\delta(y), y]v + y[\delta(v), y] = \delta(y)[v, y] = 0$ for all $v, y \in \mathcal{S}$. Again replacing v by vw, we obtain $\delta(y)v[w, y] = 0$ for all $y, v, w \in \mathcal{S}$. Since $\mathcal{S}$ is prime MA-semiring, for each $y \in \mathcal{S}$, either $\delta(y) = 0$ or $[y, w] = 0$ for all $w \in \mathcal{S}$. If $y = 0$ then obviously $y \in Z(\mathcal{S})$. Suppose $y \neq 0$ but $\delta(y) = 0$. Now, $\delta([y, s]) = \delta(ys + s'y) = \delta(s)y + s\delta(y) + \delta(y)s' + y\delta(s') = \delta(s)y + s\delta(y) + \delta(y)s' + y(\delta(s))' = \delta(s)y + s\delta(y) + \delta(y)s' + y'\delta(s)$ for all $s \in \mathcal{S}$. Since $\delta(y) = 0$, $\delta([y, s]) = \delta(s)y + y'\delta(s) = [\delta(s), y]$ for all $s \in \mathcal{S}$. Using Equation (6), we get

$$\delta([y, s]) = 0 \tag{7}$$

for all $s \in \mathcal{S}$. Again replacing s by $s[y, v]$, we get $\delta([y, s[y, v]]) = 0$ for all $s, v \in \mathcal{S}$. Thus, $\delta(s[y, [y, v]] + [y, s][y, v]) = \delta([y, [y, v]])s + [y, [y, v]]\delta(s) + \delta([y, v])[y, s] + [y, v]\delta([y, s]) = 0$ for all $v, s \in \mathcal{S}$. Using Equation (7), we get $[y, [y, v]]\delta(s) = 0$ for all $v, s \in \mathcal{S}$. Again replacing s by sr, we get $[y, [y, v]](\delta(r)s + r\delta(s)) = [y, [y, v]]\delta(r)s + [y, [y, v]]r\delta(s) = 0$ for all $v, s, r \in \mathcal{S}$. Using the last relation, we obtain $[y, [y, v]]r\delta(s) = 0$ i.e. $[y, [y, v]]\mathcal{S}\delta(s) = 0$ for all $v, s \in \mathcal{S}$. Since δ is non-zero derivation, there exists a non-zero element s_1 such that $\delta(s_1) \neq 0$. Then $[y, [y, v]]\mathcal{S}\delta(s_1) = 0$ for all $v \in \mathcal{S}$. Since $\mathcal{S}$ is prime MA-semiring, $[y, [y, v]] = 0$ for all $v \in \mathcal{S}$. Using result 2.1, we get $[y, v] = 0$ for all $v \in \mathcal{S}$. Hence, $y \in Z'(\mathcal{S}) \subset Z(\mathcal{S})$. If $y \neq 0$ and $\delta(y) \neq 0$, then $y \in Z(\mathcal{S})$ clearly. So, all cases we get $y \in Z(\mathcal{S})$. Hence $\mathcal{S}$ is commutative. $\square$

Remark 3.9. In Proposition 3.2 [3], the authors considered a generalized reverse derivation (D, δ) on a prime MA-semiring S and found that for any element $a \in S$, if $\delta(a) \neq 0$ and $[D(u), a] = 0$ for all $u \in S$, then a lies in the center of S. On the other hand, in the above theorem we have established that for any element a in a 2-torsion free prime MA-semiring $\mathcal{S}$ (which is not a ring), if $[a, \delta(a)] = 0$ (where δ is a non-zero reverse derivation) then each element of $\mathcal{S}$ lies in its centre, without putting the constraint $\delta(a) \neq 0$. Consequently, it is evident that the above theorem does not come out as a corollary of Proposition 3.2 [3].

Corollary 3.10. *Let $\delta(\neq 0)$ be a reverse derivation on a 2-torsion free prime MA-semiring $\mathcal{S}$. For each $a \in \mathcal{I}$, if $[a, \delta(a)] = 0$, where $\mathcal{I}$ is a non-zero MA-ideal of $\mathcal{S}$ but not a sub-ring of $\mathcal{S}$, then $\mathcal{S}$ is commutative.*

Proof. Using a similar approach as shown in the above Theorem, and in view of the fact that in $\delta(I)$ also becomes non-zero here, we obtain the result. □

Now we try to extend Theorems 2.1, 2.3, 2.5 and 2.6 of [5] in the setting of 2-torsion free prime MA-semiring considering generalized reverse derivation instead of generalized derivation.

Theorem 3.11. *Let $\mathcal{S}$ be a prime MA-semiring and $\mathcal{I}$ be a non-zero MA-ideal of $\mathcal{S}$. Suppose $\mathcal{S}$ admits a non-zero generalized reverse derivation F which is related with a non-zero reverse derivation δ, such that $\delta(Z'(\mathcal{S})) \neq \{0\}$. If $\mathcal{S}$ meets one of the following criteria:*

(i) $[F(ab) + a'b, s] = [F(ab) + ab', s] = 0,$

(ii) $[F(ab) + b'a, s] = [F(ab) + ba', s] = 0,$

(iii) $[F(a)F(b) + a'b, s] = [F(a)F(b) + ab', s] = 0,$

(iv) $[F(a)F(b) + b'a, s] = [F(a)F(b) + ba', s] = 0,$

for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$, then $\mathcal{S}$ is commutative.

Proof. Since $\delta(Z'(\mathcal{S}))$ is non-zero, there exists a non-zero element u in $Z'(\mathcal{S})$ such that $\delta(u) \neq 0$. So, it follows that $\delta(u) \in Z'(\mathcal{S})$ from Lemma 3.1. Since $Z'(\mathcal{S}) \subset Z(\mathcal{S})$, $\delta(u) \in Z(\mathcal{S})$.

First, assume that $[F(ab) + a'b, s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. Then we get

$$[F(b)a + b\delta(a) + a'b, s] = 0 \tag{8}$$

for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. Substituting ub in place of b, we obtain

$$[F(b)ua + b\delta(u)a + ub\delta(a) + a'ub, s] = 0 \tag{9}$$

for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. Since $\mathcal{S}$ is additively commutative, we have

$$[u(F(b)a+b\delta(a)+a'b)+b\delta(u)a, s] = u[F(ab)+a'b, s]+[b\delta(u)a, s] = 0 \quad (10)$$

for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. By hypothesis, we get $[b\delta(u)a, s] = 0$ for all $s \in S$, for all $a, b \in I$. Since $\delta(u) \in Z'(\mathcal{S})$, we get $\delta(u)[ba, s] = 0$ and hence, $\delta(u)\mathcal{S}[ba, s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. Since $\mathcal{S}$ is a prime MA-semiring and $\delta(u) \neq 0$, the equation implies that $[ba, s] = 0$ for all $a, b \in \mathcal{I}$, $s \in \mathcal{S}$. In particular, putting $s = b$ it follows that $b[a, b] = 0$ for all $a, b \in \mathcal{I}$. Hence, the required result follows from Corollary 3.5.

In a manner very similar to this, we can show that $\mathcal{S}$ becomes commutative if (ii) holds.

Now suppose $[F(a)F(b) + a'b, s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. In a similar way as in the previous proof (cf. equations (9) to (10)), we obtain $[F(a)b, s]\delta(u) = 0$ for all $a, b \in \mathcal{I}$, and for all $s \in \mathcal{S}$. Again replacing b by $F(a)b$, we get the following

$$0 = F(a)[F(a)b, s]\delta(u) + [F(a), s]F(a)b\delta(u) = [F(a), s]F(a)b\delta(u) \quad (11)$$

for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. So, from (11) $[F(a), s]F(a)b\mathcal{S}\delta(u) = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. As $\delta(u) \neq 0$, $[F(a), s]F(a)b = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Since $\mathcal{I}$ is a non-zero ideal, let us choose a non-zero element $b_1 \in \mathcal{I}$ then $[F(a), s]F(a)\mathcal{S}b_1 = 0$ for all $a \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Since $\mathcal{S}$ is a prime MA-semiring, we have $[F(a), s]F(a) = 0$ for all $a, \in \mathcal{I}$, for all $s \in \mathcal{S}$. Again replacing s by st, it follows that $[F(a), s]tF(a) = 0$ for all $a \in \mathcal{I}$, for all $s, t \in \mathcal{S}$. So, $[F(a), s]\mathcal{S}F(a) = 0$ for all $a \in \mathcal{I}$, for all $s \in \mathcal{S}$. As $\mathcal{S}$ is a prime MA-semiring, for any $a \in \mathcal{I}$ and $s \in \mathcal{S}$, either $[F(a), s] = 0$ or $F(a) = 0$. Suppose $F(a) = 0$ then clearly $F(a) = [F(a), s] = 0$ for all $s \in \mathcal{S}$. Again suppose $F(a) \neq 0$ then also $[F(a), s] = 0$ for all $s \in \mathcal{S}$. Hence, $[F(a), s] = 0$ for all $a \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Then using hypothesis, we get $[F(a)F(b) + a'b, s] = F(a)[F(b), s] + [F(a), s]F(b) + [a'b, s] = 0$ for all $s \in \mathcal{S}$ and for all $a, b \in \mathcal{I}$. Hence, $[a'b, s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$. As a result, for all $a, b \in \mathcal{I}$, $[ab, s] = 0$ for all $s \in \mathcal{S}$. In particular, putting $s = a$ follows that $a[a, b] = 0$ for all $a, b \in \mathcal{I}$, Thus, $\mathcal{S}$ is commutative according to Corollary 3.6.

Similarly, we can demonstrate that $\mathcal{S}$ becomes commutative if the condition (iv) holds. $\square$

We immediately derive the following Corollary.

Corollary 3.12. *Let $\mathcal{S}$ be a prime MA-semiring and $\mathcal{I}$ be its non-zero MA-ideal. Suppose $\mathcal{S}$ admits a non-zero generalized reverse derivation F that connected to a non-zero reverse derivation δ such that $\delta(Z'(\mathcal{S})) \neq \{0\}$. If $\mathcal{S}$ meets any of the following criterion:*

(i) $[F(ab) + ab, s] = 0,$
(ii) $[F(ab) + ba, s] = 0,$
(iii) $[F(a)F(b) + ab, s] = 0,$
(iv) $[F(a)F(b) + ba, s] = 0$

for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$, then $\mathcal{S}$ is commutative.

Now, in the next theorem, we observe what happens if we remove the condition that $\delta(Z'(\mathcal{S})) \neq \{0\}$ from the 2-torsion free prime MA-semiring under consideration.

Theorem 3.13. *Let $\mathcal{I}$ be a non-zero MA-ideal of a 2-torsion free prime MA-semiring $\mathcal{S}$ which is not a sub-ring. $\mathcal{S}$ admits a non-zero generalized reverse derivation F associated with a non-zero reverse derivation δ. If $\mathcal{S}$ satisfies any of the following conditions*

(i) $F(ab) + a'b = 0,$
(ii) $F(ab) + b'a = 0,$
(iii) $F(a)F(b) + a'b = 0,$
(iv) $F(a)F(b) + b'a = 0$

for all $a, b \in \mathcal{I}$, then $\mathcal{S}$ is commutative.

Proof. (i) First suppose that $F(ab) + a'b = 0$ for all $a, b \in \mathcal{I}$. Substituting bz in place of b, we obtain

$$F(bz)a + bz\delta(a) + a'bz = 0 \tag{12}$$

for all $a, b, z \in \mathcal{I}$. Using the given hypothesis and by (1), it follows that

$$bza + bz\delta(a) + a'bz = [bz, a] + bz\delta(a) = 0 \tag{13}$$

for all $a, b, z \in \mathcal{I}$. In particular, putting $a = b$ it follows that

$$b[z, b] + bz\delta(b) = 0 \tag{14}$$

for all $a, b, z \in \mathcal{I}$. Again replacing z by zb, we get $b[z, b]b + bzb\delta(b) = 0$ for all $b, z \in \mathcal{I}$. Multiplying the Equation (14) from the right by b and using (1), we obtain

$$bz[b, \delta(b)] = 0 \tag{15}$$

for all $b, z \in \mathcal{I}$. So, $b\mathcal{S}z[b, \delta(b)] = 0$ for all $b \in \mathcal{I}$. Thus, the primeness of $\mathcal{S}$ implies that for each $b \in \mathcal{I}$, either $b = 0$ or $z[b, \delta(b)] = 0$. Suppose $b = 0$. Then $z[b, \delta(b)] = 0$ for all $z \in \mathcal{I}$. Thus, $z[b, \delta(b)] = 0$ for all $z, b \in \mathcal{I}$. Since $\mathcal{I}$ is a non-zero MA-ideal, let us choose a non-zero element z_1. Then $z_1[b, \delta(b)] = 0$ for all $b \in \mathcal{I}$. Since $\mathcal{S}$ is a prime MA-semiring, we have $[b, \delta(b)] = 0$ for all $b \in \mathcal{I}$. Using Corollary 3.10, we get the required result.

(ii) Now suppose that $F(ab) + b'a = 0$ for all $a, b \in \mathcal{I}$. Using a similar approach as shown in the proof of *(i)* (*cf.* formulation of Equation (12) and (13)), we obtain

$$zba + b'za + bz\delta(a) = 0 \tag{16}$$

for all $a, b, z \in \mathcal{I}$. Again replacing b by br, we get $zbra + b'rza + brz\delta(a) = 0$ for all $a, b, z, r \in \mathcal{I}$. Again replacing z by rz in Equation (16), we obtain $rzba + b'rza + brz\delta(a) = 0$ for all $a, z, b, r \in \mathcal{I}$. Combining the last two equations, we have $[zb, r]a = 0$ for all $z, b, r, a \in \mathcal{I}$. Since $\mathcal{I}$ is non-zero MA-ideal, let us choose a non-zero element a_1. Then $[zb, r]\mathcal{S}a_1 = 0$ for all $z, b, r \in \mathcal{I}$. Since $\mathcal{S}$ is prime MA-semiring, we have $[zb, r] = 0$ for all $z, b, r \in \mathcal{I}$. In particular putting $z = r$, we get $r[b, r] = 0$ for all $r, b \in \mathcal{I}$. Then using Corollary 3.5, we get $\mathcal{S}$ is commutative.

(iii) By our assumption, we get $F(a)F(b) + a'b = 0$ for all $a, b \in \mathcal{I}$. Replacing b by zb and using the given hypothesis, we obtain

$$a[b, z] + F(a)b\delta(z) = 0 \tag{17}$$

for all $a, b, z \in \mathcal{I}$. Replacing b by bz, it follows $a[b, z]z + F(a)bz\delta(z) = 0$ for all $a, b, z \in \mathcal{I}$. In the equation (17) by multiplying on the right side by z and by (1), we arrive at $F(a)b[\delta(z), z] = 0$ for all $a, b, z \in \mathcal{I}$. So, $F(a)\mathcal{S}b[\delta(z), z] = 0$ for all $a, b, z \in \mathcal{I}$. Suppose $F(a) = 0$ for all $a \in \mathcal{I}$. Then from hypothesis, we get $a'b = 0$ i.e. $ab = 0$ for all $a, b \in \mathcal{I}$. Hence, $a\mathcal{S}b = 0$ for all $a, b \in \mathcal{I}$. This is impossible in view of the fact $\mathcal{I}$ is non-zero and $\mathcal{S}$ is a prime MA-semiring. So, there exists a non-zero element $a_1 \in \mathcal{I}$ such that $F(a_1) \neq 0$. Then $F(a_1)\mathcal{S}b[\delta(z), z] = 0$ for all $b, z \in \mathcal{I}$. Since $\mathcal{S}$ is a prime MA-semiring, $b[\delta(z), z] = 0$ for all $b, z \in \mathcal{I}$. As $\mathcal{I}$ is non-zero and $\mathcal{S}$ is a prime MA-semiring, $[\delta(z), z] = 0$ for all $z \in \mathcal{I}$. Using Corollary 3.10, we get $\mathcal{S}$ is commutative.

(iv) By given hypothesis $F(a)F(b) + b'a = 0$ for all $a, b \in \mathcal{I}$. Substitute sb in place of b, we get

$$[ba, s] + F(a)b\delta(s) = 0 \tag{18}$$

for all $a, b, s \in \mathcal{I}$. Again replacing b with sb, we obtain $s[ba, s]+F(a)sb\delta(s) = 0$ for all $a, b, s \in \mathcal{I}$. Multiplying by a in Equation (18) from the left side and using (1), we arrive at $[F(a), s]b\delta(s) = 0$. This implies $[F(a), s]b\mathcal{S}\delta(s) = 0$ for all $a, b, s \in \mathcal{I}$. Let $t \in \mathcal{I}$. Then by primeness of $\mathcal{S}$, we get either $[F(a), t]b = 0$ or $\delta(t) = 0$. If $t = 0$ then obviously $[F(a), t]b = \delta(t) = 0$ for all $a, b \in \mathcal{I}$. Suppose $t \neq 0$ but $\delta(t) = 0$. Then from Equation (18), we get $[ba, t] = 0$ for all $a, b \in \mathcal{I}$. Replace a by ar, we get $ba\mathcal{S}[r, t] = 0$ for all $a, b, r \in \mathcal{I}$. By primeness of $\mathcal{S}$ either we get $ba = 0$ or $[r, t] = 0$ for all $a, b, r \in \mathcal{I}$. Since $\mathcal{I}$ is non-zero, $ab = 0$ is not possible for all $a, b \in \mathcal{I}$. Let us choose a_1, $b_1 \in \mathcal{I}$ such that $b_1a_1 \neq 0$. Then $b_1a_1\mathcal{S}[r, t] = 0$ for all $r \in \mathcal{I}$. By primeness of $\mathcal{S}$, we get $[r, t] = 0$ for all $r \in \mathcal{I}$. Suppose $t \neq 0$ and $\delta(t) \neq 0$. Then $[F(a), t]b = 0$ i.e. $[F(a), t]\mathcal{S}b = 0$ for all $a, b \in \mathcal{I}$. Since $\mathcal{I}$ is non-zero and $\mathcal{S}$ is prime, we have $[F(a), t] = 0$ for all $a \in \mathcal{I}$. Replacing a by xa, we get

$$[F(xa), t] = [F(a)x, t] + [a\delta(x), t] = 0 \tag{19}$$

for all $a, x \in \mathcal{I}$. In particular, for $x = t$, we obtain $[a\delta(t), t] = 0$ for all $a, \in \mathcal{I}$. Again replacing a by wa, we get $[w, t]a\delta(t) = 0$ for all $a, w \in \mathcal{I}$. Since $\delta(t) \neq 0$ and $\mathcal{S}$ is a prime MA-semiring, we get $[w, t]a = 0$ i.e. $[w, t]\mathcal{S}a = 0$ for all $w, a \in \mathcal{I}$. Since $\mathcal{I}$ is non-zero and $\mathcal{S}$ is prime, we obtain $[w, t] = 0$ for all $w \in \mathcal{I}$. For any choice $t \in \mathcal{I}$, we get $[r, t] = 0$ for all $r \in \mathcal{I}$. Using Lemma 3.4, we get $\mathcal{S}$ is commutative. □

Theorem 3.14. *Let $\mathcal{S}$ be a prime MA-semiring. Suppose F is a non-zero generalized reverse derivation associated with a non-zero reverse derivation δ such that $F(ab) + [a, b] = 0$ or $F(ab) + a \circ b = 0$ for all $a, b \in \mathcal{I}$ where $\mathcal{I}$ is a non-zero MA-ideal which is not a subring. Then $\mathcal{S}$ is commutative.*

Proof. We assume that $F(ab) + [a, b] = 0$ for all $a, b \in \mathcal{I}$. Replacing b by bz yields that $F(bz)a + bz\delta(a) + abz + bza' = 0$ for all $a, b, z \in \mathcal{I}$. Using hypothesis and by (1), it follows

$$2bza' + zba + abz + bz\delta(a) = 0 \tag{20}$$

for all $a, b, z \in \mathcal{I}$. Substituting ab instead of b, we obtain

$$2abza' + zaba + aabz + abz\delta(a) = 0 \tag{21}$$

for all $a, b, z \in \mathcal{I}$. Multiplying by a from the left side of Equation (20), it gives $2ab'za + azba + aabz + abz\delta(a) = 0$ for all $a, b, z \in \mathcal{I}$. Combining the last two relations, we get $[z, a]ba = 0$ i.e. $[z, a]b\mathcal{S}a = 0$ for all $a, b, z \in \mathcal{I}$.

Suppose $a \neq 0$. Then primeness of $\mathcal{S}$ gives $[z,a]b = 0$ for all $b, z \in \mathcal{I}$. Thus, $[z,a]b = 0$ for all $a, b, z \in \mathcal{I}$, whence $[z,a]\mathcal{S}b_1 = 0$ for all $a, z \in \mathcal{I}$ and some non-zero element b_1 of $\mathcal{I}$. Again using primeness of $\mathcal{S}$, we get $[z,a] = 0$ for all $a, z \in \mathcal{I}$. Hence, $\mathcal{S}$ is commutative using Lemma 3.4.
From given hypothesis, we obtain $F(ab) + a \circ b = 0$ for all $a, b \in \mathcal{I}$. Using a similar approach as shown in the first part of the proof (cf. formulation of Equation (20)), we get $bza' + zba' + abz + bza + bz\delta(a)$ for all $a, b, z \in \mathcal{I}$. This implies

$$abz + zba' + bz\delta(a) = 0 \tag{22}$$

for all $a, b, z \in \mathcal{I}$. Again replacing z by rz, we get $abrz + rzba' + brz\delta(a) = 0$ for all $a, b, z, r \in \mathcal{I}$. Again replacing b by br in Equation (22), we get $abrz + zbra' + brz\delta(a) = 0$ for all $a, b, z, r \in \mathcal{I}$. Combining the last two equations, we get $[r, zb]a' = 0$ for all $a, b, z, r \in \mathcal{I}$. Since $\mathcal{I}$ is non-zero MA-ideal, $[r, zb]\mathcal{S}a_1 = 0$ for all $r, b, z \in \mathcal{I}$ and for some non-zero $a_1 \in \mathcal{I}$. By primeness of $\mathcal{S}$, we get $[r, zb] = 0$ for all $r, b, z \in \mathcal{I}$. In particular, putting $z = r$ we get $r[r, b] = 0$ for all $r, b \in \mathcal{I}$. Using Corollary 3.6, we get $\mathcal{S}$ is commutative. □

We derive the following theorems using a similar approach as illustrated in the preceding Theorem.

Theorem 3.15. *Let $\mathcal{S}$ be a prime MA-semiring and $\mathcal{I}$ be a non-zero MA-ideal of $\mathcal{S}$. Suppose $\mathcal{S}$ admits a non-zero generalized reverse derivation F associated with a non-zero reverse derivation δ such that $\delta(Z'(\mathcal{S})) \neq \{0\}$. If $\mathcal{S}$ satisfies either $[F(ab) + [a,b], s] = 0$ or $[F(ab) + a \circ b, s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$, then $\mathcal{S}$ is commutative.*

Proof. By replacing b by ub we can easily obtain that $[b\delta(u)a, s] = 0$ for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$ and $u \in Z'(\mathcal{S})$. Hence by proceeding in a similar manner as shown in the proof of Theorem 3.11 (i) we obtain the result. □

Theorem 3.16. *Let $\mathcal{S}$ be a 2-torsion free prime MA-semiring and $\mathcal{I}$ be a non-zero MA-ideal which is not a subring of $\mathcal{S}$. If $\mathcal{S}$ admits a non-zero generalized reverse derivation F associated with a non-zero reverse derivation δ such that $F[a,b] + [a,b] = 0$ or $F(a \circ b) + a \circ b = 0$ for all $a, b \in \mathcal{I}$, then $\mathcal{S}$ is commutative.*

Proof. For any $a, b \in \mathcal{I}$, we have $F([a,b]) + [a,b] = 0$. Substituting ab in place of b, we get $F(a[a,b]) + a[a,b] = 0$ for all $a, b \in \mathcal{I}$. Utilizing the hypothesis, this leads to the relation

$$[a, [a, b]] + [a, b]\delta(a) = 0 \tag{23}$$

for all $a, b \in \mathcal{I}$. Replacing b by ba, it gives

$$[a, [a, b]]a + [a, b]a\delta(a) = 0 \tag{24}$$

for all $a, b \in \mathcal{I}$. Multiplying in Equation (23) by a from the right side , we get

$$[a, [a, b]]a + [a, b]\delta(a)a = 0 \tag{25}$$

for all $a, b \in \mathcal{I}$. Comparing Equations (24) and (25), we obtain $[a, b][a, \delta(a)] = 0$ for all $a, b \in \mathcal{I}$. Substituting $\delta(a)b$ in place of b, we get $[a, \delta(a)]b[a, \delta(a)] = 0$ for all $a, b \in \mathcal{I}$. Thus, we get $[a, \delta(a)]\mathcal{S}b[a, \delta(a)] = 0$ for all $a, b \in \mathcal{I}$. Since $\mathcal{I}$ is a non-zero MA-ideal, let us choose a non-zero element b_1. Then $[a, \delta(a)]\mathcal{S}b_1[a, \delta(a)] = 0$ for all $a \in \mathcal{I}$. By primeness of $\mathcal{S}$, we get $[a, \delta(a)] = 0$ for all $a \in \mathcal{I}$. Using Corollary 3.10, we get $\mathcal{S}$ is commutative.
For any $a, b \in \mathcal{I}$ we have $F(a \circ b) + a \circ b = 0$. By definition of generalized reverse derivation

$$F(b)a + b\delta(a) + F(a)b + a\delta(b) + ab + ba = 0 \tag{26}$$

for all $a, b \in \mathcal{I}$. Replacing b with ab, we get

$$F(b)aa + b\delta(a)a + ab\delta(a) + F(a)ab + a\delta(b)a + ab\delta(a) + aab + aba = 0 \tag{27}$$

for all $a, b \in \mathcal{I}$. Multiplying form right side by a in Equation (26), comparing the above equation, we get

$$2ab\delta(a) + F(a)[a, b] + b'aa + aab = 0 \tag{28}$$

for all $a, b \in \mathcal{I}$. Replacing b by ba, we get $2aba\delta(a) + F(a)[a, b]a + b'aaa + aab = 0$ for all $a, b \in \mathcal{I}$. Multiplying in Equation (28) by a from the right side, we get $2ab\delta(a)a + F(a)[a, b]a + b'aaa + aaba = 0$ for all $a, b \in \mathcal{I}$. Combining the last two equations, we get $2ab[a, \delta(a)] = 0$ for all $a, b \in \mathcal{I}$. Since $\mathcal{S}$ is a 2-torsion free prime MA-semiring, $ab[a, \delta(a)] = 0$ for all $a, b \in \mathcal{I}$, which is similar as the Equation (15) of the Theorem 3.13 (i). Now arguing similarly, we conclude that $\mathcal{S}$ is commutative. □

Theorem 3.17. *Let $\mathcal{S}$ be a 2-torsion free prime MA-semiring and $\mathcal{I}$ be a non-zero MA-ideal of $\mathcal{S}$. Suppose $\mathcal{S}$ admits a non-zero generalized reverse derivation F associated with a non-zero reverse derivation δ such that $\delta(Z'(\mathcal{S})) \neq \{0\}$. If $\mathcal{S}$ satisfies any of the following conditions*

(*i*) $[F[a,b] + [a,b], s] = 0$,

(*ii*) $[F(a \circ b) + a \circ b, s] = 0$

for all $a, b \in \mathcal{I}$, for all $s \in \mathcal{S}$, then $\mathcal{S}$ is commutative.

Proof. As, $\delta(Z'(\mathcal{S}))$ is non-zero, there exists an element $u \in Z'(\mathcal{S})$ such that $\delta(u) \neq 0$. Therefore, from Lemma 3.1, it follows that $\delta(u) \in Z'(\mathcal{S}) \subset Z(\mathcal{S})$. First, we assume that $[F[a,b]+[a,b], s] = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Replacing b by ub, we get $\delta(u)[[a,b], s] = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Hence, $\delta(u)[[a,b], b] = 0$ for all $a, b \in \mathcal{I}$. Since $\delta(u) \neq 0$ and $\mathcal{S}$ is a prime MA-semiring, we get $[[a,b], b] = 0$ for all $a, b \in \mathcal{I}$. Using Lemma 3.7, we get $\mathcal{S}$ is commutative.

Suppose that $[F(a \circ b) + a \circ b, s] = 0$ for all $a, b \in \mathcal{I}$ and $s \in \mathcal{S}$. Replacing b by ub, we get $u[F(a \circ b) + a \circ b, s] + [b\delta(u)a + ab\delta(u), s] = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Using the hypothesis, we get $[b\delta(u)a + ab\delta(u), s] = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Since $\delta(u) \in Z(\mathcal{S})$, we obtain $[a \circ b, s]\delta(u) = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Again replacing s by ts we get $[a \circ b, s]t\delta(u) = 0$ for all $a, b \in \mathcal{I}$ and for all $s, t \in \mathcal{S}$. As $\delta(u) \neq 0$ and $\mathcal{S}$ is a prime MA-semiring, we have $[a \circ b, s] = 0$ for all $a, b \in \mathcal{I}$ for all $s \in \mathcal{S}$. Again replacing b by ba, we get $[a \circ (ba), s] = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Hence, $(a \circ b)[a, s] = 0$ for all $a, b \in \mathcal{I}$ and for all $s \in \mathcal{S}$. Again replacing s by rs, we get $(a \circ b)r[a, s] = 0$ for all $a, b \in \mathcal{I}$ and for all $r, s \in \mathcal{S}$. Again replacing b by bc, we get $(a \circ bc)r[a, s] = (b(a \circ c) + [a, b]c)r[a, s] = 0$ for all $a, b, c \in \mathcal{I}$ and for all $r, s \in \mathcal{S}$. Thus, $[a, b]cr[a, s] = 0$ for all $a, b, c \in \mathcal{I}$ and for all $r, s \in \mathcal{S}$. In particular, putting $s = b$, we obtain $[a, b]cr[a, b] = 0$ for all $a, b, c \in \mathcal{I}$ and for all $r \in \mathcal{S}$. Let c_1 be a non-zero element of $\mathcal{I}$. Then $[a, b]c_1\mathcal{S}[a, b] = 0$ for all $a, b \in \mathcal{I}$. By primeness of $\mathcal{S}$, we obtain $[a, b] = 0$ for all $a, b \in \mathcal{I}$. Using Lemma 3.4, we get $\mathcal{S}$ is commutative. □

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