Quasigroups and Related Systems **34** (2026), 21 – 30

https://doi.org/10.56415/qrs.v34.02

Nil extensions of Clifford ordered semigroups

Anjan Kumar Bhuniya and Kalyan Hansda

Abstract. Following Cao [11], an ideal extension S of I is called a nil extension of I if $(S/I, \cdot, \preceq)$ is a nil ordered semigroup. In this paper, we consider the characterizations of ordered semigroups which are the nil extension of some class of regular ordered semigroups, such as, (left) group like ordered semigroups, (left) Clifford ordered semigroups.

1. Introduction

Nil extensions of semigroups, without order, are precisely the result of ideal extensions by a nil semigroup. This concept was first explored in 1984 by S. Bogdanović and S. Milić [9], who characterized semigroups without order that qualify as nil extensions of completely simple semigroups. A similar investigation was conducted by J. L. Galbiati and M. L. Veronesi [12] in 1980. Subsequently, S. Bogdanović and M. Cirić [5] delved into the study of nil extensions across various types of semigroups, including regular semigroups, groups, periodic semigroups, and completely regular semigroups. Further research on nil extensions of semigroups was documented in [4], [6], and [7].

The concept of ideal extensions in ordered semigroups was introduced by N. Kehayopulu and M. Tsingelis [15]. Y. Cao [11] expanded this notion to ordered semigroups, specifically examining nil extensions of simple ordered semigroups and their complete semilattice decompositions. Additionally, N. Kehayopulu and M. Tsingelis [14] provided a characterization of ordered semigroups that serve as nil extensions of Archimedean ordered semigroups.

Motivated by the works of S. Bogdanovic and M. Ciric [5], [8], our paper explores the realm of Clifford (left Clifford) ordered semigroups, a subclass within the domain of regular ordered semigroups introduced in [1]. The present study examines the key characteristics of ordered semigroups which are nil extensions of Clifford, Clifford ordered semigroups and some other subclasses of regular ordered semigroups.

The structure of our paper is as follows: Section 2 presents the fundamental definitions and properties of ordered semigroups, while Section 3 delves into the characterization of nil extensions of Clifford and left Clifford ordered semigroups.

2010 Mathematics Subject Classification: 20M10; 06F05.

Keywords: nil extension, regular ordered semigroups, group-like, left group-like, Clif–ford and left Clifford ordered semigroups.

2. Preliminaries

In this paper, $\mathbb{N}$ denotes the set of all natural numbers. An *ordered semigroup* S is a partially ordered set $(S, \leqslant)$, and at the same time a semigroup $(S, \cdot)$ such that for all $a,\ b,\ x \in S$,

$$a \leqslant b \text{ implies } xa \leqslant xb \text{ and } ax \leqslant bx.$$

It is denoted by $(S, \cdot, \leqslant)$. We follow Birkhoff [2] to define the zero element of an ordered semigroup. An element 0 in S is called a *zero* of S if $0 \leqslant x$ and $0x = x0 = 0$ for every $x \in S$. Thus zero of S is unique (if it exists) and it is the zero of the semigroup $(S, \cdot)$ as well as the least element of the partially ordered set $(S, \leqslant)$. Cao and Xinzhai [11] defined zero element of an ordered semigroup in a different way. An ordered semigroup S with 0 is called *nilpotent* if for every $a \in S$ there is $n \in \mathbb{N}$ such that $a^n = 0$.

For an ordered semigroup S and $H \subseteq S$, denote the *downward closure* by:

$$(H] := \{t \in S : t \leqslant h, \text{ for some } h \in H\}.$$

Let I be a nonempty subset of an ordered semigroup S. Then I is a *left* (*right*) *ideal* of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. We call I an *ideal* of S if I is both a left and a right ideal of S. A (*left, right*) *ideal* I of S is proper if $I \neq S$. An ordered semigroup S is called (*left, right*) *simple* if it has no proper (left, right) ideals. The principal [13] left ideal, right ideal and ideal generated by $a \in S$ are denoted by $L(a), R(a)$ and $I(a)$, respectively. It is easy to check that

$$L(a) = (a \cup Sa],\ R(a) = (a \cup aS], \text{ and } I(a) = (a \cup Sa \cup aS \cup SaS].$$

The intersection of all ideals of an ordered semigroup S, if it is nonempty, is called the *kernel* of S and is denoted by $K(S)$.

An element $a \in S$ is called *ordered regular* if $a \leqslant axa$ for some $x \in S$. If every element of S is ordered regular then S is called a *regular ordered semigroup*. We call S *left regular* (resp. *right regular*) if for every $a \in S,\ a \in (Sa^2]$ ($(a^2S]$). An ordered semigroup S is called a *group-like ordered semigroup* [1] if for all $a, b \in S$ there are $x, y \in S$ such that $a \leqslant xb$ and $a \leqslant by$. Every group like ordered semigroup is a regular ordered semigroup. An ordered semigroup S is called a *left group-like ordered semigroup* [1] if S is a regular ordered semigroup and for all $a, b \in S$ there is $x \in S$ such that $a \leqslant xb$. Right group-like ordered semigroup are defined dually. A regular ordered semigroup S is called a *Clifford ordered semigroup* (resp. *left Clifford ordered semigroup*) [1] if for all $a, b \in S$ there is $x \in S$ such that $ab \leqslant bxa$ ($ab \leqslant xa$). We denote the set of all regular, left regular, right regular of an ordered semigroup S by $Reg_{\leqslant}(S)$, $\mathbf{L}Reg_{\leqslant}(S)$ and $\mathbf{R}Reg_{\leqslant}(S)$.

In an ordered semigroup S, an equivalence relation ρ is said to be *congruence* if for $a, b, c \in S,\ a\ \rho\ b$ implies $ca\ \rho\ cb$ and $ac\ \rho\ bc$. A congruence ρ on S is a called *semilattice congruence* if for all $a, b \in S,\ a\ \rho\ a^2$ and $ab\ \rho\ ba$. A semilattice congruence ρ on S is called *complete* if $a \leqslant b$ implies $(a, ab) \in \rho$, in other words if there exists a semilattice Y and a family of subsemigroups $\{S_\alpha\}_{\alpha \in Y}$ of S of some type such that:

1. $S_\alpha \cap S_\beta = \phi$ for any $\alpha,\ \beta \in Y$ *with* $\alpha \neq \beta$,
2. $S = \bigcup_{\alpha \in Y} S_\alpha$,
3. $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for any $\alpha,\ \beta \in Y$,
4. $S_\beta \cap (S_\alpha] \neq \phi$ implies $\beta \preceq \alpha$, where $\preceq$ is the order of the semilattice Y defined by

$$\preceq := \{(\alpha,\ \beta)\ |\ \alpha = \alpha\,\beta\,(\beta\,\alpha)\}\ [17].$$

Let I be an ideal of S. Set $S/I = (S\backslash I) \cup \{0\}$, where 0 is an arbitrary element of I ($S\backslash I$ is the complement of K *to* S). Define an operation "$*$" and an order "$\preceq$" on S/I as follows:

$$x * y = \begin{cases} xy & \text{if} \quad xy \in S\backslash I \\ 0 & \text{if} \quad xy \in I \end{cases}$$

and $\preceq = (\leqslant \cap[(S\backslash I) \times (S\backslash I)]) \cup \{(0, x) | x \in S/I\}$. Kehayopulu and Tsingelis [15] proved the following result:

Lemma 2.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and I be an ideal of S. Then $(S/I, *, \preceq)$ is an ordered semigroup and 0 is the zero element of S/I.*

For the sake of simplicity we use the same notation $\leqslant$ to represent order of every ordered semigroup.

Note that $(S/I, *)$ is a Rees semigroup where the congruence is the Rees congruence on S. Kehayoupulu and M. Tsingelis [15] defined ideal extension of an ordered semigroup as follows:

Definition 2.2. [15] Let $(S, \cdot, \leqslant_S)$ be an ordered semigroup, $(Q, \cdot, \leqslant_Q)$ an ordered semigroup with 0, $S \cap Q^* = \phi$, *where* $Q^* = Q \setminus \{0\}$. An ordered semigroup $(V, \cdot, \leqslant_V)$ is called an ideal extension of S by Q if there exists an ideal S' of V such that $(S', \cdot, \leqslant_{S'}) \approx (S, \cdot, \leqslant_S)$ and $(V/S', *, \preceq) \approx (Q, \cdot, \leqslant_Q)$, where $\leqslant_{S'} = \leqslant_V \cap (S' \times S')$ and "$*$", "$\preceq$" the multiplication and the order on V/S' defined above.

Kehayopulu and Tsingelis [Theorem 1, [15]] proved that for given an ordered semigroup S and an ordered semigroup Q with 0, there is an ordered semigroup V which is an ideal extension of S by Q and conversely each extension V of S by Q, can be so constructed.

The following results are useful.

Lemma 2.3. [11] *Let S be an ordered semigroup and I an ideal of S. Then the following conditions are equivalent:*
1. *S is a nil-extension of I;*
2. *For all $a \in S$ there exists $m \in \mathbb{N}$) $a^m \in I$.*

Theorem 2.4. [1] *Let S be a regular ordered semigroup. Then followings hold in S:*
1. *S is Clifford if and only if $\mathcal{L} = \mathcal{R}$.*
2. *$\mathcal{L}$ is a complete semilattice congruence if S is Clifford.*
3. *S is Clifford ordered semigroup if and only if it is a complete semilattice of group-like ordered semigroups.*

Theorem 2.5. [1] *Let S be a regular ordered semigroup. Then followings hold in S:*

1. *$\mathcal{L}$ is a complete semilattice congruence if S is left Clifford.*
2. *S is left Clifford ordered semigroup if and only if it is a complete semilattice of left group-like ordered semigroups.*

3. Main Results

Here we characterize ordered semigroups which are the nil extensions of (left) Clifford, (left) group-like ordered semigroups.

We omit the proof of the following lemma as it is straightforward.

Lemma 3.1. *A regular ordered semigroup S is a left group-like ordered semigroup if and only if $a \in (aSab]$ if for all $a,\ b \in S$.*

Theorem 3.2. *An ordered semigroup S is a nil extension of a left group-like ordered semigroup if and only if for every $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in (a^n S a^n b]$ and for every $a \in S,\ b \in Reg_{\leqslant}(S),\ a \leqslant ba$ implies $a \leqslant axb$ for some $x \in S$.*

Proof. First suppose that S is a nil extension of a left group-like ordered semigroup K and $a, b \in S$. Then there is $m \in \mathbb{N}$ such that $a^m \in K$. Regularity of K implies that $a^m \leqslant a^m x a^m$ for some $x \in K$. Further, for $xa^m, a^m b \in K$; the left simplicity of K yields that $xa^m \leqslant ya^m b$, for some $y \in K$. This gives that $a^m \leqslant a^m x a^m \leqslant a^m y a^m b$.

Next let $b \in Reg_{\leqslant}(S)$ and $a \in S$ such that $a \leqslant ba$. Since $b \in Reg_{\leqslant}(S)$, there exists $z \in S$ such that $b \leqslant b(zb)^n$ for all $n \in \mathbb{N}$. Then for some $n_1 \in \mathbb{N}$, $(zb)^{n_1} \in K$; whence $b(zb)^{n_1} \in K$. Since K is an ideal of S, $b \in K$ and so $ba \in K$. Thus $a \in K$. Since K is a left group-like ordered semigroup, for $a, ba \in K$ it follows that $a \leqslant asba$ for some $s \in K$, by Lemma 3.1. Thus the given conditions follow.

Conversely, assume that given conditions hold in S. Let $a \in S$. Then by given condition we have $a^m \leqslant a^m x a^{m+1}$, for some $x \in S$ and $m \in \mathbb{N}$. This implies $a^{m+1} \in Reg_{\leqslant}(S)$ and so $Reg_{\leqslant}(S) \neq \phi$. Denote $T = Reg_{\leqslant}(S)$.

Let $s \in S$ and $a \in T$. Then the definition of T implies $a \leqslant a(xa)^n$, for all $n \in \mathbb{N}$ and some $x \in S$. Thus $sa \leqslant sa(xa)^n$, for all $n \in \mathbb{N}$. Now for $xa, sa \in S$, there exists $m_1 \in \mathbb{N}$ and $t_1 \in S$ such that $(xa)^{m_1} \leqslant (xa)^{m_1} t_1 (xa)^{m_1} sa$, by the first condition. Then $a \leqslant a(xa)^{m_1}$ implies $sa \leqslant sa(xa)^{m_1}$ and hence $sa \leqslant sa(xa)^{m_1} t_1 (xa)^{m_1} sa$; where $(xa)^{m_1} t_1 (xa)^{m_1} \in S$. So $sa \in T$. Also $a \leqslant (ax)^n a$ for all $n \in \mathbb{N}$. Let $m_2 \in \mathbb{N}$ be such that $(ax)^{m_2} \in T$. Then $as \leqslant (ax)^{m_2} as$ implies by the second condition that

$$as \leqslant ast_2(ax)^{m_2} \text{ for some } t_2 \in S. \tag{1}$$

Denote $t = (ax)^{m_2}$. Then the definition of T implies

$$t \leqslant t(zt)^n \text{ for all } n \in \mathbb{N} \text{ for some } z \in T. \tag{2}$$

Using the first condition, for $zt, as \in S$ we have $(zt)^{m_3} \leqslant (zt)^{m_3} t_3 (zt)^{m_3} as$ for some $m_3 \in \mathbb{N}$ and $t_3 \in S$. That is $t \leqslant t(zt)^{m_3} t_3 (zt)^{m_3} as$, by (2). So from (1) we have $as \leqslant ast_2 t(zt)^{m_3} t_3 (zt)^{m_3} as$, and hence $as \in T$.

Next choose $a \in S$ and $b \in T$ such that $a \leqslant b$. Since $b \in T$ there is $x \in S$ such that $b \leqslant b(xb)^n$ for all $n \in \mathbb{N}$. Now for $xb, a \in S$, it follows from the first condition that

$$(xb)^{m_4} \leqslant (xb)^{m_4} t_4 (xb)^{m_4} a \text{ for some } m_4 \in \mathbb{N} \text{ and } t_4 \in S. \tag{3}$$

Then $a \leqslant b$ implies $a \leqslant b(xb)^{m_4} t_4 (xb)^{m_4} abt_5 a$; where $t_5 = (xb)^{m_4} t_4 (xb)^{m_4}$.

Since $b \in T$, by above we have $bt_5 \in T$. Say $bt_5 = t_6$. Then for $a \in S$, and $t_6 \in T$, $a \leqslant t_6 a$ yields that $a \leqslant at_7 t_6$ for some $t_7 \in S$, by second condition. Therefore $a \leqslant at_7 bt_5 = at_7 b(xb)^{m_4} t_4 (xb)^{m_4} \leqslant at_7 b(xb)^{m_4} t_4 (xb)^{m_4} t_4 (xb)^{m_4} a$, by (3).

Clearly $t_7b(xb)^{m_4}t_4(xb)^{m_4}t_4(xb)^{m_4} \in T$, as $b \in T$. Thus $a \in T$, which shows that T is an ideal of S.

Finally consider $c, d \in T$. Then there is $v \in S$ such that $c \leqslant c(vc)^n$ for all $n \in \mathbb{N}$. Now for $vc, d \in S$, there exists $t_8 \in S$ such that $c \leqslant c(vc)^{m_5}t_8(vc)^{m_5}d$ for some $m_5 \in \mathbb{N}$. Since $c \in T$, $c(vc)^{m_5}t_8(vc)^{m_5} \in T$. Hence T is left simple. Thus T is left group-like ordered semigroup such that for every $a \in S$ there is $m \in \mathbb{N}$, $a^m \in T$. Hence S is nil extension of a left group-like ordered semigroup T. □

In the following result we provide an independent proof of Corollary 5.2 of [11].

Theorem 3.3. *An ordered semigroup S is a nil extension of a group-like ordered semigroup if and only if for all $a, b \in S$, there exists $n \in \mathbb{N}$ such that $a^n \in (b^nSb^n]$.*

Proof. Suppose that S is a nil extension of a group-like ordered semigroup G and $a, b \in S$. Then there exists $n \in \mathbb{N}$ such that $a^n, b^n \in G$. Since G is a group like ordered semigroup, there exists $u \in G$ such that $a^n \leqslant b^nu$. Also for $u, b^n \in G$ there exists $x \in G$ such that $u \leqslant xb^n$. This implies $b^nu \leqslant b^nxb^n$. Thus $a^n \leqslant b^nxb^n$ and hence $a^n \in (b^nSb^n]$.

Conversely, let us assume that given condition holds in S. Choose $a \in S$. Then for some $m \in \mathbb{N}$ and $x \in S$, $a^m \leqslant a^mxa^m$. Thus $Reg_{\leqslant}(S) \neq \phi$. Say $G = Reg_{\leqslant}(S)$. So for every $a \in S$, there exists $m \in \mathbb{N}$ such that $a^m \in G$. Let us consider $b \in G$ and $s \in S$. Then for all $n \in \mathbb{N}$

$$bs \leqslant bybs \leqslant (by)^nbs \text{ for } y \in S. \tag{4}$$

Using the given condition for $bs, by \in S$, we obtain $(by)^mbs \leqslant (bs)^mz(bs)^{m+1}$ for $z \in S$ and $m \in \mathbb{N}$. This yields that

$$\begin{aligned} bs &\leqslant (bs)^mz(bs)^{m+1} \\ &\leqslant bstbs; \text{ where } t = bs^{m-1}zbs^m. \end{aligned}$$

Thus $bs \in G$. Similarly $sb \in G$.

Next let $a \in S$ and $b \in G$ be such that $a \leqslant b$. Since $b \in G$ there exists $x \in S$ such that $b \leqslant bxb$ and hence $b \leqslant (bx)^nb(xb)^n$ for all $n \in \mathbb{N}$, which implies that

$$\begin{aligned} a &\leqslant a^m(z_1a^mba^nz)a^n, \text{ for some } m, n \in \mathbb{N} \text{ and } z, z_1 \in S \\ &= ata, \text{ where } t = a^{m-1}z_1a^mba^nza^{n-1}. \end{aligned}$$

So $a \in G$. Hence G is an ideal of S.

Finally, consider $a, b \in G$. Then there exists $x \in S$ such that

$$a \leqslant (ax)^na \text{ for all } n \in \mathbb{N},$$

and so by the given condition it follows that $a \leqslant b^{m'}z'b^{m'}a$ for some $m' \in \mathbb{N}$ and $z' \in S$. This gives that $a \leqslant bu$ for some $u = b^{m'-1}z'b^{m'}a \in G$. Similarly there is some $v \in S$ such that $a \leqslant vb$. This shows that G is a group-like ordered semigroup. Hence S is a nil extension of a group-like ordered semigroup G. □

Theorem 3.4. *An ordered semigroup S be a nil extension of a Clifford ordered semigroup if and only if for every $x, a, y \in S$, there exists $n \in \mathbb{N}$ such that $xa^ny \in (xa^nySya^nx] \cap (ya^nxSxa^ny]$ and $a \in S$, $b \in Reg_{\leqslant}(S)$ such that $a \leqslant b$, implies $a \in (Sab]$.*

Proof. First let S is a nil extension of a Clifford ordered semigroup K. Let $x, a, y \in S$. Then there is $m \in \mathbb{N}$ such that $a^m \in K$. Since K is an ideal of S, $xa^m y \in K$. Since K is a regular, there exists $z_1 \in K$ such that $xa^m y \leqslant xa^m yz_1xa^m y$. For $z_1x, a^m y \in K$ implies that

$$z_1xa^m y \leqslant (a^m y)u_1(z_1x), \text{ for some } u_1 \in K, \text{ since } S \text{ Clifford}. \tag{5}$$

Similarly for $a^m, (yu_1z_1) \in K$ there is $u_2 \in S$ such that

$$a^m(yu_1z_1) \leqslant (yu_1z_1)u_2a^m. \tag{6}$$

Therefore

$$\begin{aligned} xa^m y &\leqslant xa^m yz_1xa^m y \\ &\leqslant xa^m ya^m(yu_1z_1)x, \text{ by (5)} \\ &\leqslant xa^m y^2u_1z_1u_2a^m x, \text{ by (6)}. \end{aligned}$$

Thus

$$xa^m y \leqslant xa^m yz_1xa^m y^2u_1z_1u_2a^m x. \tag{7}$$

Also, for $a^m y^2, u_1z_1u_2 \in K$ there exists $u_3 \in K$ such that $a^m y^2u_1z_1u_2 \leqslant u_1z_1u_2u_3a^m y^2$. Then from (7), we obtain that

$$\begin{aligned} xa^m y &\leqslant xa^m y(z_1xu_1z_1u_2u_3a^m y)ya^m x \\ &\leqslant xa^m ysya^m x; \text{ where } s = z_1xu_1z_1u_2u_3a^m y. \end{aligned}$$

Therefore $xa^m y \in (xa^m ySya^m x]$. Similarly $xa^m y \in (ya^m xSxa^m y]$.

Now $K \subseteq Reg_{\leqslant}(S)$ implies that $Reg_{\leqslant}(S) \neq \phi$. Consider $b \in Reg_{\leqslant}(S)$ and $a \in S$ such that $a \leqslant b$. Since $b \in Reg_{\leqslant}(S)$, there exists $z \in S$ such that $b \leqslant (bz)^n b$ for all $n \in \mathbb{N}$. Since S is a nil extension of K, there exists $n_1 \in \mathbb{N}$ such that $(bz)^{n_1} \in K$. This gives $(bz)^{n_1}b \in K$, which gives $b \in K$ and so $a \in K$, as K is an ideal of S. Further K is a Clifford ordered semigroup, so by Theorem 2.4 $\mathcal{L}$ is a congruence on S. Since $a, b \in K$ we have $a\mathcal{L}ab$ and hence $a \in (Sab]$.

Conversely assume that given conditions hold in S. Let $a \in S$ be arbitrary. Then by the first condition there exists $n \in \mathbb{N}$ such that $a^{n+2} \leqslant a^{n+2}xa^{n+2}$, for some $x \in S$. Thus $Reg_{\leqslant}(S) \neq \phi$. Say $T = Reg_{\leqslant}(S)$. It is now clear that for each $a \in S$, there exists $m \in \mathbb{N}$ such that $a^m \in T$.

Let $s \in S$ and $x \in T$. Then for all $n \in \mathbb{N}$ and for some $t \in S$, $x \leqslant (xt)^n x$. This implies that $sx \leqslant sx(tx)^{n-1}tx$, for all $n \in \mathbb{N}$. By first condition there are $s_1 \in S$ and $m_1 \in \mathbb{N}$ such that $sx \leqslant sx(tx)^{m_1}txs_1tx(tx)^{m_1}sx$ and thus $sx \leqslant sxpsx$; where $p = (tx)^{m_1}txs_1tx(tx)^{m_1}$. Also for every $n \in \mathbb{N}$, $xs \leqslant x(tx)^n s \leqslant xt(xt)^{n-1}xs$. So there is $m_2 \in \mathbb{N}$ such that $xs \leqslant xs(xt)^{m_2}xts_2xt(xt)^{m_2}xs \leqslant xsqxs$; where $q = (xt)^{m_2}xts_2xt(xt)^{m_2}$. Thus $sx, sx \in T$.

To show T is a Clifford ordered semigroup, choose $a, b \in T$. Then there is $r \in S$ such that

$$\begin{aligned} ab &\leqslant abrab \\ &\leqslant (abra)(bra)^{n-1}b, \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{8}$$

Now for $abra, bra, b \in S$, the first condition yields that

$$abra(bra)^{m_3}b \leqslant b(bra)^{m_3}abrap_1abra(bra)^{m_3}b \text{ for some } p_1 \in S \text{ and } m_3 \in \mathbb{N}.$$

Therefore from (8) we have

$$\begin{aligned} ab &\leqslant b(bra)^{m_3}abrap_1abra(bra)^{m_3}b \\ &\leqslant b(bra)^{m_3}abrap_1abra(bra)^{m_3-1}brab \\ &\leqslant bgab; \text{ where } g = (bra)^{m_3}abrap_1abra(bra)^{m_3-1}br \in T. \end{aligned} \tag{9}$$

Similarly there are $m_4 \in \mathbb{N}$ and $p_2 \in S$ such that $ab \leqslant abra(bra)^{m_4}bp_2b(bra)^{m_4}\ abra$. So from (9), $ab \leqslant b(gabra(bra)^{m_4}bp_2b(bra)^{m_4})abra = bua$; where $u = gabra(bra)^{m_4}bp_2b(bra)^{m_4}abr \in T$. Hence T is Clifford ordered semigroup.

Now let $a \leqslant b$ for some $a \in S$ and $b \in T$. Then by the second condition, there is $z \in S$ such that $a \leqslant zab$, that is,

$$a \leqslant zabtab \quad \text{for some } t \in S. \tag{10}$$

Since T is Clifford ordered semigroup, for $zabt, ab \in T$ it follows that $zabtab \leqslant abp_3zabt$ for some $p_3 \in S$. Similarly for $bp_3za, bt \in T$, we have $bp_3zabt \leqslant btp_4bp_3za$ for some $p_4 \in S$. The last two inequalities together with (10) yields that $a \leqslant aha$, where $h = btp_4bp_3z$. Thus $a \in T$ and so T is an ideal of S. Hence S is a nil extension of a Clifford ordered semigroup T. □

Theorem 3.5. *An ordered semigroup S is a nil extension of a left Clifford ordered semigroup if and only if for every $x, a, y \in S$, there exists $n \in \mathbb{N}$ such that $xa^ny \in (xa^nySya^nx] \cap (xa^nySxa^ny]$ and for $a \in S$, $b \in Reg_{\leqslant}(S)$ such that $a \leqslant b$ implies $a \leqslant azab$ for some $z \in S$.*

Proof. Let S be a nil extension of a left Clifford ordered semigroup K. Choose $x, a, y \in S$. Then there exists $m \in \mathbb{N}$ such that $a^m \in K$. Since K is an ideal of S, $xa^my \in K$. Also the regularity of K yields that

$$xa^my \leqslant xa^myz_1xa^my, \text{ for some } z_1 \in K. \tag{11}$$

Since K is a left Clifford ordered semigroup and $z_1x, a^my \in K$, we have $z_1xa^my \leqslant z_2(z_1x)$ for some $z_2 \in K$. Therefore from (11),

$$\begin{aligned} xa^my &\leqslant xa^myz_1xa^my \\ &\leqslant xa^myz_1xa^myz_1xa^my \\ &\leqslant xa^myz_1xa^myz_2(z_1x). \end{aligned} \tag{12}$$

Similarly, for $a^m, yz_2z_1 \in K$ there is $z_3 \in K$ such that

$$a^myz_2z_1 \leqslant z_3a^m, \tag{13}$$

and for $a^m, z_1xz_3 \in K$ there is $z_4 \in K$ such that

$$a^myz_1xz_3 \leqslant z_4a^my. \tag{14}$$

Thus from (12) we obtain that

$$\begin{aligned} xa^m y &\leqslant xa^m yz_1xa^m yz_2z_1x \\ &\leqslant xa^m yz_1xa^m z_3a^m x, \text{ from (13)} \\ &\leqslant xa^m yz_4a^m ya^m x, \text{ from (14)}. \end{aligned}$$

Hence $xa^m y \in (xa^m ySya^m x]$ and so $xa^m y \in (xa^m ySya^m x] \cap (xa^m ySxa^m y]$, from (11). For second condition let $a \in S$ and $b \in Reg_{\leqslant}(S)$ be such that $a \leqslant b$. Then $b \leqslant b(tb)^n$ for some $t \in S$ and for all $n \in \mathbb{N}$. Then there is $r \in \mathbb{N}$ such that $(tb)^r \in K$. Since K is an ideal $b(tb)^r \in K$ and so $b \in K$. Thus $a \in K$. Since K is left Clifford ordered semigroup, $\mathcal{L}$ is congruence on S, by Theorem 2.5. So for $a, b \in K$, $a\mathcal{L}ab$ and thus from Theorem 3.1, $a \leqslant azab$ for some $z \in S$. This proves the necessary condition.

Conversely, suppose that given conditions hold in S. Let $a \in S$. Then there is $n \in \mathbb{N}$ such that $a^{n+2} \leqslant a^{n+2}xa^{n+2}$, for some $x \in S$, by the first condition. Thus $Reg_{\leqslant}(S) \neq \phi$. Say $T = Reg_{\leqslant}(S)$. Now for each $a \in S$, there exists $m \in \mathbb{N}$ such that $a^m \in T$.

Let $s \in S$ and $x \in T$. Then for all $n \in \mathbb{N}$ and for some $t \in S$, $x \leqslant (xt)^n x$, which gives $sx \leqslant sx(tx)^{n-1}tx$, for all $n \in \mathbb{N}$.

By the first condition there are $s_1 \in S$ and $m_1 \in \mathbb{N}$ such that

$$\begin{aligned} sx &\leqslant sx(tx)^{m_1}txs_1tx(tx)^{m_1}sx \\ &\leqslant sxpsx; \text{ where } p = (tx)^{m_1}txs_1tx(tx)^{m_1}. \end{aligned}$$

Therefore $sx \in T$.

We now show that T is a left Clifford ordered semigroup. For this let us assume that $a, b \in T$. Then there is $t_1 \in S$ such that $ab \leqslant abt_1ab \leqslant a(bt_1a)^n bt_1ab$ for all $n \in \mathbb{N}$. Then by first condition, there are $t_2 \in S$ and $m' \in \mathbb{N}$ such that

$$\begin{aligned} ab &\leqslant a(bt_1a)^{m'}bt_1ab \\ &\leqslant a(bt_1a)^{m'}bt_1abt_2bt_1ab(bt_1a)^{m'}a \end{aligned}$$

Therefore $ab \leqslant t_1'a$; where $t_1' = a(bt_1a)^{m'}bt_1abt_2bt_1ab(bt_1a)^{m'}$. Since $a \in T$ we have $t_1' \in T$. Hence T is a left Clifford ordered semigroup.

Next let $a \in S$ and $b \in T$ such that $a \leqslant b$. Using second condition we have $a \leqslant azab$, for some $z \in S$. Then for all $n \in \mathbb{N}$, $a \leqslant a(za)^n b^n$. Now for some $m'' \in \mathbb{N}$, $(za)^{m''} \in T$. Since T is a left Clifford, $(za)^{m''}b^{m''} \in (Ta]$. So $a \in (aSa]$, that is $a \in T$.

Finally to show T, an ideal of S we need only to show that $xs \in T$. The regularity of x yields that $xs \leqslant x(tx)^n s$ for all $n \in \mathbb{N}$. For $x, tx, s \in S$ there is $l \in \mathbb{N}$ such that $x(tx)^l s \in (x(tx)^l sSx(tx)^l s]$, by given condition. Thus $l \in \mathbb{N}$, $x(tx)^l s \in T$. Then $xs \leqslant x(tx)^l s$ implies that $xs \in T$, by above. Thus T is an ideal of S. Hence S is a nil extension of a left Clifford ordered semigroup T. □

Acknowledgements. The authors thank the Editor-in-Chief, Prof. Wieslaw A. Dudek, and the anonymous reviewers for their careful reading of the manuscript and for their helpful comments and suggestions, which have significantly improved the quality, clarity, and presentation of this work.

References

[1] **A.K. Bhuniya and K. Hansda**, *On completely regular and Clifford ordered semigroups*, Afrika Mat., **31** (2020), 1029-1045.

[2] **G. Birkhoff**, Lattice Theory, Amer. Math. Soc. Coll. Publ., Providence, 1967.

[3] **S. Bogdanović**, Semiroups with a system of subsemigroups, Inst. Math., Univ. Novi Sad, 1985.

[4] **S. Bogdanović and M. Cirić**, *Semigroups of Galbiati-Veronesi- III*, Facta Univ. (Nis) Ser. Math. Inform., **4** (1989), 1-14.

[5] **S. Bogdanović and M. Cirić**, *A nil extension of a regular semigroup*, Glasnik Mat., **25(2)** (1991), 3-23.

[6] **S. Bogdanović and M. Cirić**, *Retractive nil-extensions of regular semigroups-I*, Proc. Japan Acad. Math. Sci., **68** (1992), 115-117.

[7] **S. Bogdanović and M. Cirić**, *Retractive nil-extensions of regular semigroups-II*, Proc. Japan Acad. Math. Sci., **68** (1992), 126-130.

[8] **S. Bogdanović and M. Cirić**, *Semilattices of nil extension of simple regular semigroup*, Algebra Coll., **10(1)** (2003), 81-90.

[9] **S. Bogdanović and S.Milić**, *A nil extension of completely simple semigroup*, Publ. Inst. Math. (Beograd), **36(50)** (1984), 45-50.

[10] **Y. Cao**, *On weak commutativity of po-semigroups and their semilattice decompositions*, Semigroup Forum, **58** (1999), 386-394.

[11] **Y. Cao and X. Xinzhai**, *Nil-extensions of simple po-semigroups*, Commun. Algebra, **28** (2000), 2477-2496.

[12] **J.L. Galbiati and M.L. Veronesi**, *Sui semiguppi che sono un band di t-semigruppi*, Instituto Lombardo (Rend. Sc)-A, **114** (1980), 217-234.

[13] **N. Kehayopulu**, *Ideals and Green's relations in ordered semigroups*, Intern. J. Math. and Math. Sci., (2006), 1-8, Article ID 61286.

[14] **N. Kehayopulu**, *Archimeadean ordered semigroups as ideal extensions*, Semigroup Forum, **78** (2009), 343-348.

[15] **N. Kehayopulu and M. Tsingelis**, *Ideal extensions of ordered semigroups*, Commun. Algebra, **31** (2003), 4939-4969.

[16] **N. Kehayopulu and M. Tsingelis**, *Ordered semigroups having P- property*, Russian Math. (Iz. VUZ), **52(12)** (2008), 23-27.

[17] **N. Kehayopulu and M. Tsingelis**, *Semilattices of Archimedean ordered semigroups*, Algera Coll., **15** (2008), 527-540.

Received October 15, 2025

Department of Mathematics
Visva Bharati University
Santiniketan, Bolpur - 731235
West Bengal, India
e-mail:anjankbhuniya@gmail.com, kalyanh4@gmail.com