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Topological semigroups from quasi-pseudometrics and quasi-norms: an asymmetric generalization

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Abstract. We explore topologies on semigroups and groups induced by families of quasi-pseudometrics and quasi-norms, aiming to transform them into topological semigroups and groups. We establish conditions under which these topologies are compatible with the algebraic operations of semigroups and groups. Finally, we demonstrate the potential of these topologies to endow a group into a bitopological group structure.

1. Introduction

The problem of constructing topologies on algebraic structures that are compatible with their operations is a fundamental one in topology and algebra. A classical method for achieving this involves using families of distance functions. The pseudometric space, originally introduced by Kuratowski, generalizes the concept of metric space. Another significant and valuable generalization is the concept of a quasi-pseudometric space, introduced and developed by Kelly J.C.[5]. A pivotal result in this area is due to Markov A.A. [7], who showed how to constructed a topology on groups using norms. This was later expanded upon by Mukhin V.V and Boujouf H. [2], who established a characterization theorem for semigroups and groups. Their work showed the equivalence between the joint continuity of multiplication, the continuity of right translations, and a pointwise uniform condition for families of symmetric pseudometrics and norms. In recent years, the study of non-symmetric topology has gained considerable attention due to its applications in theoretical computer science and applied physics. This shift brings structures such as quasi-pseudometrics and quasi-norms to the forefront, naturally raising a fundamental question: *do the classical compatibility results still hold when the assumption of symmetry is abandoned?*

In this work, we answer this question affirmatively. We investigate topologies on semigroups and groups generated by families of left-invariant, right-invariant, and invariant quasi-pseudometrics, pseudometrics, quasi-norms, and norms. Theorem 2.5 is a non symmetric generalization of the result in [2]. We prove that the

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fundamental equivalence between joint continuity, continuity of tanslations and a pointwise condition holds for families of left-invariant quasi-pseudometrics. To provide a powerful tool for applying this theory we introduce the concept of a right X-closed family(Condition D), which offers a verifiable sufficient condition for a topology to be a semigroup topology.

A key insight of this paper is that the theory for quasi-norms is not a separate endeavor but rather a direct and elegant application of our main results on quasi-pseudometrics. By viewing a quasi-norm N as generating a left-invariant quasi-pseudometric $f_N(x,y) = N(y^{-1}x)$, we seamlessly translate problems about quasi-norms into our established framework. This approach allows us to efficiently derive a complete characterization of group topologies generated by families of quasi-norms (Theorem 2.8).

Finally, we demonstrate that these topologies can transform a group into a bitopological group. We examine the relationship between the natural left and right topologies induced by quasi-norms, providing conditions under which they together form a bitopological structure.

A mapping $f : X \times X \to [0, +\infty)$ is called a *quasi-pseudometric* on a set X if for all $x, y, z \in X$, $f(x,x) = 0$ and $f(x,y) \leqslant f(x,z) + f(z,y)$. Additionally, if $f(x,y) = f(y,x)$ then f is said to be a *pseudometric*. A quasi-pseudometric (or the pseudometric) f on a semigroup X is said to be *left-invariant* (resp. *right-invariant*) if $f(xy, xz) = f(y,z)$ $(resp. f(yx, zx) = f(y,z))$ for all $x, y, z \in X$. Furthermore, f is called *invariant* if it is both left-invariant and right-invariant.

A semigroup (group) X is called a topological semigroup (topological group) if it is endowed with a Hausdorff topology and the multiplication $(x,y) \mapsto xy : X \times X \to X$ is jointly continuous (and in the case of a group, the inversion $x \mapsto x^{-1} : X \to X$ is also continuous).

If a group X is endowed with two topologies τ_1 and τ_2, where the mappings $(x,y) \mapsto xy : (X,\tau_1) \times (X,\tau_2) \to (X,\tau_1)$; $(x,y) \mapsto xy : (X,\tau_1) \times (X,\tau_2) \to (X,\tau_2)$; $x \mapsto x^{-1} : (X,\tau_1) \to (X,\tau_1)$; $x \mapsto x^{-1} : (X,\tau_2) \to (X,\tau_2)$ are continuous, then $(X, \cdot, \tau_1, \tau_2)$ is called a *bitopological group*.

Let $(G, \cdot)$ be a group. A function $N : G \to [0, +\infty)$ is called a *quasi-norm* if $N(e) = 0$,where e is the unit element of the group G, and there exists a constant $K \geqslant 1$ such that $N(xy) \leqslant K(N(x) + N(y))$ for all x, y from G. If, in addition, $N(x) = N(x^{-1})$ then the quasi-norm is called a *norm*.

For undefined notation and terminologies on bitopological spaces, we refer the interested reader to [5].

2. Results

A family Φ of quasi-pseudometrics on a set X generates a topology τ_f on X. A subbase for this topology is given by the sets $B_f(a,\epsilon) = \{x \in X : f(a,x) < \epsilon\}$,for any $f \in \Phi$, $a \in X$, and $\epsilon > 0$.

Theorem 2.1. *Let a topology τ_f on a semigroup X be generated by a family of quasi-pseudometrics Φ.*

(1) *If each $f \in \Phi$ is left-invariant, then for any $z \in X$ the left translation $\lambda_z : X \to X, z \mapsto zx$ is continuous in (X, τ_f).*

(2) *If each $f \in \Phi$ is right-invariant, then for any $z \in X$ the right translation $\rho_z : X \to X, z \mapsto xz$ is continuous in (X, τ_f).*

Proof. (1). We show that the preimage of any subbasic open set $U = B_f(a, \epsilon)$ is open. For this let $x_0 \in \lambda_z^{-1}(U)$, so $f(a, zx_0) < \epsilon$. Let $\delta = \epsilon - f(a, zx_0) > 0$. Consider the neighborhood $V = B_f(x_0, \delta)$. For any $x \in V$, $f(x_0, x) < \delta$. By left invariance, $f(zx_0, zx) = f(x_0, x) < \delta$. By the triangle inequality, $f(a, zx) \leqslant f(a, zx_0) + f(zx_0, zx) < f(a, zx_0) + \delta = \epsilon$. This shows $x \in \lambda_z^{-1}(U)$, so $V \subseteq \lambda_z^{-1}(U)$. Since x_0 was arbitrary, $\lambda_z^{-1}(U)$ is open and it follows that λ_z is continuous in (X, τ_f).

The proof of (2) is analogous, using the right-invariance property. □

Theorem 2.2. *Let a topology τ_f on a semigroup X be generated by a family of invariant pseudometrics $\{f\}$. Then, for any fixed $a, b \in X$ the two-sided translation map $\eta_{a,b} : X \to X, x \mapsto axb$ is continuous in (X, τ_f) .*

Proof. The map $\eta_{a,b}$ can be written as the composition $\eta_{a,b}(x) = \rho_b(\lambda_a(x))$. By Theorem 2.1, both ρ_b and λ_a are continuous. Since the composition of continuous functions is continuous, it follows that $\eta_{a,b}$ is also continuous. □

Corollary 2.3. *Under the same conditions as in Theorem 2.2, the central translation map $\eta_z : X \to X, x \mapsto zxz$ is continuous.*

Proof. This is the special case of Theorem 2.2 with $a = b = z$. □

Remark 2.4. Let τ be a topology on a set X generated by the family Φ of quasi-pseudometrics; that is, τ is the topology induced by the uniformity $\mathcal{U}_\Phi$, which has as a base the entourages $\{(x, y) \in X \times X : f(x, y) < \epsilon\}$, $f \in \Phi$, $\epsilon > 0$.

It follows from the definition of a uniform space that different families of quasi-pseudometrics may generate the same topology. In particular, if ψ is another family of quasi-pseudometrics such that the uniformities $\mathcal{U}_\Phi$ and $\mathcal{U}_\Psi$ coincide, then Ψ also generates τ. Consequently, when studying properties of the topological structure (X, τ) that depend only on the topology (and not on the specific choice of a generating family), we are free to replace Φ by any uniformly equivalent family. This flexibility is used in the proof of Theorem 2.3, where we construct a right $X-$closed family Φ^* that generates the same topology as the original family Φ.

Theorem 2.5. *Let $(X, \cdot)$ be a semigroup, and let τ_f be the topology on X generated by the family Φ of left-invariant quasi-pseudometrics. The following conditions are equivalent:*

(A) $(X, \cdot, \tau)$ *is a topological semigroup.*

(B) *For every $a \in X$, the right translations $\rho_a : X \to X, x \mapsto xa$ are continuous.*

(C) *For every $a \in X$, every $f \in \Phi$, and every $\epsilon > 0$, there exist $g_1, \ldots, g_n \in \Phi$ and a number $\delta > 0$ such that for all $x, y \in X$*
$$\max_{1 \leqslant i \leqslant n} g_i(y, x) < \delta \Rightarrow f(xa, ya) < \epsilon.$$

(D) *The family Φ is right X-closed, i.e. for every $f \in \Phi$, and every $z \in X$, the function $f_z(x, y) = f(xz, yz)$ is also belongs to Φ.*

(E) *For every $a \in X$, the right translation map $\rho_a \colon (X, \mathcal{U}_\Phi) \to (X, \mathcal{U}_\Phi)$ is uniformly continuous, where $\mathcal{U}_\Phi$ is the left-invariant uniformity generated by Φ.*

Furthermore, if X is a group and $f(x, e) = f(e, x)$ for all $x \in X$, where e is the identity, then any of the obove conditions implies that the inversion map $x \mapsto x^{-1}$ is continuous.

Proof. We establish the equivalences by demonstrating the cycle $(D) \Rightarrow (C) \Leftrightarrow (E) \Rightarrow (A) \Rightarrow (B) \Rightarrow (C) \Rightarrow (D)$.

$(D) \Rightarrow (C)$: If Φ is right X-closed, then for any $a \in X$, any $f \in \Phi$, the function $f_a(x, y) = f(xa, ya)$ belongs to Φ. Taking $g_1 = f_a$ and $\delta = \epsilon$, we obtain (C).

$(C) \Leftrightarrow (E)$: This follows directly from the definition of uniform continuity of the right translation ρ_a with respect to the uniformity generated by Φ.

$(E) \Rightarrow (A)$: Fix $a, b \in X$, $f \in \Phi$, and $\epsilon > 0$. By (E), there exist $g_1, \ldots, g_n \in \Phi$ and a number $\delta > 0$ such that $\max_{1 \leqslant i \leqslant n} g_i(y, x) < \delta \Rightarrow f(xb, yb) < \epsilon$. Using left-invariance, if x is sufficiently close to a and y is sufficiently close to b, then xy is close to ab. This proves joint continuity of multiplication at (a, b) i.e., $(X, \cdot, \tau)$ is a topological semigroup.

$(A) \Rightarrow (B)$: If multiplication is continuous, then for any fixed $a \in X$, the map $x \mapsto xa = \rho_a(x)$ is continuous, as it is the composition of the continuous map $x \mapsto (x, a)$ and the continuous multiplication.

$(B) \Rightarrow (C)$: Assume all right translations ρ_a are continuous. Let $a \in X$, $\epsilon > 0$, and $f \in \Phi$. For a fixed $x \in X$ The set $B_f(xa, \epsilon) = \{z \in X : f(xa, z) < \epsilon\}$ is $\tau-$open neighborhood of $xa = \rho_a$. By the continuity of ρ_a, the preimage $\rho_a^{-1}(B_f(xa, \epsilon))$ is $\tau-$open neighborhood of x. Since τ is generated by Φ, there exist $g_1, \ldots, g_n \in \Phi$ and $\delta > 0$ such that: $\{y \in X : \max_{1 \leqslant i \leqslant n} g_i(y, x) < \delta\} \subset \rho_a^{-1}(B_f(xa, \epsilon))$. This implies that if $\max_{1 \leqslant i \leqslant n} g_i(y, x) < \delta$, then $f(xa, ya) < \epsilon$, which is condition (C).

$(C) \Rightarrow (D)$: Assume condition (C) holds. Fix $z \in X$ and $f \in \Phi$, and define the function $f_z(x, y) = f(xz, yz)$. We show that f_z belongs to the uniform closure of Φ. By (C) applied to $a = z$, for every $\epsilon > 0$ there exist $g_1, \ldots, g_n \in \Phi$ and $\delta > 0$ such that $\max_{1 \leqslant i \leqslant n} g_i(y, x) < \delta \Rightarrow f(xz, yz) < \epsilon$. This implies that the entourage $\{(x, y) \in X \times X \colon f_z(x, y) < \epsilon\}$ contains the entourage $\bigcap_{i=1}^{n} \{(x, y) \in X \times X \colon g_i(y, x) < \delta\}$. Consequently, f_z is uniformly continuous with respect to the uniformity generated by Φ. Now define $\Phi^* = \{f_z \colon f \in \Phi, z \in X\}$. Since each f_z is $\tau_\Phi-$continuous, the topology generated by Φ^* is contained in τ_Φ. Conversely, $\Phi \subseteq \Phi^*$, so $\tau_\Phi \subseteq \tau_{\Phi^*}$. Thus, Φ^* generates the same topology τ_Φ. By Remark 2.4, we may replace Φ with the uniformly equivalent family Φ^*. By construction, Φ^* is X-closed. This establishes (D).

Now, suppose X is a group and one of the equivalent conditions holds, with $f(x,e) = f(e,x)$ for all $f \in \Phi$ and $x \in X$. The inversion is continuous at the identity e, indeed, consider $\epsilon > 0$, By the joint continuity of multiplication (condition (A)) at (e,e), there exist $g_1, \ldots, g_n \in \Phi$ and $\delta > 0$ such that if $\max_i g_i(x,e) < \delta$ and $\max_i g_i(y,e) < \delta$, then $f(xy,e) < \epsilon$.Let $x \in X$ satisfy $\max_i g_i(x,e) < \delta$. Then $f(xx^{-1},e) = f(e,e) = 0 < \epsilon$. Using symmetry and left invariance, $f(x^{-1},e) = f(e,x^{-1}) = f(x,e) < \epsilon$. Hence, inversion is continuous at e. Since the topology τ is translation -invariant, inversion is continuous on all of X. This completes the proof. □

Example 2.6. 1. The semigroup $(\mathbb{N},+)$.

Define $d(x,y) = |x-y|$. Then $d_a(x,y) = |(x+a)-(y+a)| = |x-y| = d(x,y)$. Hence, the family $\{d\}$ is right $\mathbb{N}$-closed, so the induced topology is compatible with the semigroup structure.

2. The semigroup $(\mathbb{R}^+,\cdot)$.

Define $d(x,y) = |x-y|$. For $a \geqslant 0$, $d_a(x,y) = |xa - ya| = a|x-y|$, which is again a multiple of d. Thus, the family $\{d\}$ is right $\mathbb{R}$-closed, and the topology is semitopological.

3. Matrix semigroups.

For $S = M_n(\mathbb{R})$ with matrix multiplication, let $d(A,B) = \|A-B\|$ for some operator norm. Then $d_p(A,B) = \|AP-BP\| = \|(A-B)P\| \leqslant \|P\| \cdot \|A-B\|$, so the family $\{d\}$ is right S-closed. Hence, $M_n(\mathbb{R})$, endowed with the operator norm topology, is a semitopological semigroup.

Corollary 2.7. *Let Φ be a right X-closed family of left-invariant quasi-pseudometrics on a semigroup X. Then the topology τ generated by Φ makes $(X,\cdot,\tau)$ into a topological semigroup.*

Proof. This is an immediate consequence of the implication $(D) \Rightarrow (A)$ in Theorem 2.5. □

Remark 2.8. The proof of the equivalence of the conditions (A), (B), and (C) in Theorem 2.5, established via the cycle $(C) \Rightarrow (A) \Rightarrow (B) \Rightarrow (C)$, is particularly noteworthy. It does not rely on the symmetry property of the quasi-pseudometrics. The crucial step in the argument, which generalizes the proof in [2], hinges on the fact that for a function $g_j \in \Phi$, the relation $g_j(sh,sy) = g_j(h,y)$ holds as a direct consequence of left-invariance. This demonstrates that the fundamental compatibility between the algebraic and topological structures is determined by the properties of invariance and the triangle inequality, not by the symmetry of the metric.

Remark 2.9. In [2], the equivalence among conditions (A), (B), and (C) was established. The present theorem extends this result by showing that these conditions are further equivalent to (D) and (E). Thus, right-closure of the generating family and uniform continuity of right translations provide natural structural characterizations of topological semigroup topologies.

Theorem 2.10. *Let G be a group, and let a topology τ on G be generated by a family of invariant pseudometrics Φ. Then, for any $a, b \in G$, the two-sided translation map $\eta_{a,b} : G \to G$, defined by $\eta_{a,b}(x) = axb$, is a homeomorphism.*

Proof. Continuity follows from Theorem 2.2. To show it is a homeomorphism, we note its inverse is also a two-sided translation: $(\eta_{a,b})^{-1}(x) = \eta_{a^{-1},b^{-1}}(x) = a^{-1}xb^{-1}$. Since a^{-1} and b^{-1} exist in a group, and $\eta_{a^{-1},b^{-1}}$ is also continuous by Theorem 2.2, it follows that $\eta_{a,b}$ is a homeomorphism. □

A classical construction provides a canonical example of a right X-closed family, thereby illustrating the power of Corollary 2.7. Let μ be a finitely additive measure defined on a ring S of subsets of a semigroup X, which is left-invariant on S; that is, $\mu(B) = \mu(xB)$ for any B from Borel sets $B(X)$ and $x \in X$.

Theorem 2.11. *Let $S_0 \subset S$ be a subfamily such that for each $A \in S_0$, we have $\mu(A) < \infty$ and $xA \in S_0$ for all x in the semigroup X. Then the function $f_A(x, y) = \mu(xA\Delta yA)$ defines a left-invariant pseudometric on X. Furthermore, the family $\Phi = \{f_A\}_{A\in S_0}$ is right X-closed. Consequently, the topology generated by the family $\{f_A\}_{A\in S_0}$ turns $(X, \cdot, \tau)$ into a topological semigroup.*

Proof. The verification that f_A is a pseudometric is standard (see,e.g., [8, Lemma 6, p.67]). To prove left-invariance, let $x, y, a \in X$. Then

$$f_A(ax, ay) = \mu(axA\Delta ayA) = \mu(a(xA\Delta yA)) = \mu(xA\Delta yA) = f_A(x, y),$$

where the critical second equality holds because the symmetric difference commutes with the bijection $z \mapsto az$, and the third equality follows from the left-invariance of the measure μ.

Next, since for every $f_A \in \Phi$ and every $z \in X$, $(f_A)_z(x, y) = f_A(xz, yz) = \mu(xzA\Delta yzA) = f_{zA}(x, y)$, then the family $\{f_A\}_{A\in S_0}$ is right X-closed.

Consequently, by an immediate application of Corollary 2.7 the topology generated by this family turns $(X, \cdot, \tau)$ into a topological semigroup. □

The following theorem provides a powerful and elegant characterization of group topologies. It generalizes Theorem 4 from [2], by replacing the family of globally symmetric pseudometrics with a family of quasi-pseudometrics that are only required to be symmetric at the identity. This distinction is crucial, as it shows that a weaker algebrac property can still yield the same strong topological result.

Theorem 2.12. *Let G be a group and τ a topology on G. Then $(G, \cdot, \tau)$ is a topological group if and only if the topology τ is generated by a right G-closed family Φ of left invariant quasi-pseudometrics, along with $f(x,e) = f(e,x)$ for all $f \in \Phi$ and all $x \in G$.*

Proof. ($\Rightarrow$). Assume that $(G, \cdot, \tau)$ is a topological group.
1. Construction of left-invariant pseudometrics. Let $\mathcal{N}(e)$ denote the filter of neighborhoods of the identity e in G. For each $U \in \mathcal{N}(e)$, define a function $d_U : G \times G \longrightarrow [0, +\infty)$ by $d_U(x,y) = inf\{n \geqslant slant1 : xy^{-1} \in U^n\}$ where $U^1 = U$ and $U^{n+1} = U^n \cdot U$. It is a standard result (see [3],p.59) that each d_U is a left-invariant pseudometric, and the family $\Psi = \{d_U : U \in \mathcal{N}(e)\}$ generates the left uniformity and the topology τ. Since each d_U is a pseudometric, it satisfies the symmetry condition $d_U(x,e) = d_U(e,x)$ for all $x \in G$.

2. Closing under right translations. Define $\Phi = \{(d_U)_z : d_U \in \Psi, z \in G\}$, where $(d_U)_z(x,y) = (d_U)(xz, yz)$. Since each d_U is left-invariant, each $(d_U)_z$ is also left-invariant. By construction, Φ is right G-closed: for any $(d_U)_z \in \Phi$ and any $w \in G$, $((d_U)_z)_w(x,y) = d_U(xzw, yzw) = (d_U)_{zw}(x,y) \in \Phi$.

3. Φ generates τ. For any $d_U \in \Psi$, we have $d_U = (d_U)_e \in \Phi$ so the topology generated by Φ contains τ. Conversely, for any $z \in G$, the right translation $\rho_z : G \longrightarrow G$ is a homeomorphism of (G, τ). Hence, the pseudometric $(d_U)_z$ generates the same topology as d_U. Therefore, Φ generates exactly the topology τ.

4. Symmetry condition. For any $(d_U)_z \in \Phi$ and any $x \in G$, $(d_U)_z(x,e) = d_U(xz, ez) = d_U(xz, z)$, and $(d_U)_z(e,x) = d_U(ez, xz) = d_U(z, xz)$. Since d_U is a pseudometric, $d_U(xz, z) = d_U(z, xz)$. Hence, $(d_U)_z(x,e) = (d_U)_z(e,x)$, and the symmetry condition holds for the entire family Φ.
Thus, Φ is a right G-closed family of left-invariant quasi-pseudometrics satisfying the symmetry condition at the identity, and it generates τ. This establishes the necessity.

($\Leftarrow$). Assume that the topology τ is generated by a family Φ with the given properties. By Corollary 2.7, a topology generated by a right G-closed family of left-invariant quasi-pseudometrics makes G into a topological semigroup. This ensures that the multiplication map is jointly continuous. Moreover, the symmetry condition at the identity $f(x,e) = f(e,x)$ for all $f \in \Phi$ is precisely the condition required by Theorem 2.5 to guarantee that the inversion map is also continuous. Since both multipliction and inversion are continuous, $(G, \cdot, \tau)$ is a topological group. □

This section applies the general theory developed above to the important special case of topologies generated by families of quasi-norms on a group. We establish characterization theorems and explore the resulting bitopological structures. Note that every quasi-norm N, on the group G naturally induces two quasi-pseudometrics:

- The left-invariant quasi-pseudometric :$f_N(x,y) = N(y^{-1}x)$.

• The right-invariant quasi-pseudometric $g_N(x,y) = N(xy^{-1})$.

Let $\mathcal{N}$ be a family of quasi-norms on G. This family generates two natural topologies:

• The left topology τ_l, generated by the family $\Phi_l = \{f_N : N \in \mathcal{N}\}$.

• The right topology τ_r, generated by the family $\Phi_r = \{g_N : N \in \mathcal{N}\}$.

In general, τ_l and τ_r are distinct. The primary focus is on the left topology τ_l, which we will often denote simply by τ.

Note that each norm N is a continuous function with respect to both the right and left topologies.

Theorem 2.13. *Let $\mathcal{N}$ be a nonempty family of quasi-norms on the group G. Define $\Phi_l = \{f_N(x,y) = N(y^{-1}x) : N \in \mathcal{N}\}$, and let τ be a topology on G generated by Φ_l. The following conditions are equivalent:*

(A') *$(G, \cdot, \tau)$ is a topological group.*

(B') *For every $a \in G$, the right translations $\rho_a : G \to G, x \mapsto xa$ are continuous.*

(C') *For every $a \in G$, $N \in \mathcal{N}$, and $\alpha > 0$, there exist $N_1, \dots, N_k \in \mathcal{N}$ and a number $\beta > 0$ such that $\max\limits_{1 \leqslant i \leqslant k} N_i(x^{-1}s) < \beta \Rightarrow N((xa)^{-1}sa) < \alpha$ for all $s, x \in G$.*

(D') *the family $\mathcal{N}$ is conjugation-closed, i.e., for every $N \in \mathcal{N}$ and every $z \in G$, the function N_z defined by $N_z(x) = N(z^{-1}xz)$ also belongs to $\mathcal{N}$.*

(E') *For every $a \in G$, the right translation $\rho_a : (G, \mathcal{U}) \to (G, \mathcal{U})$ is uniformly continuous, where $\mathcal{U}$ is the uniformity generated by Φ_l.*

Furthermore, if each $N \in \mathcal{N}$ satisfies $N(x) = N(x^{-1})$ for all $x \in G$, then the inversion map $x \mapsto x^{-1}$ is continuous.

Proof. The equivalences follow by translating the conditions $(A) - (E)$ from Theorem 2.5 into the family Φ_l generated by $N \in \mathcal{N}$:

The equivalence of (A'), (B'), (E') is immediate from the difinitions, since $(A') \Leftrightarrow (A)$, $(B') \Leftrightarrow (B)$, $(E') \Leftrightarrow (E)$.

The equivalence of (C') with (C) follows by substituting $f = f_N$ and $g_i = f_{N_i}$ which shows that (C') is precisely (C) expressed in terms of quasi-norms.

The equivalence of (D') with (D): Assume $\mathcal{N}$ is conjugation-closed ((D')). Let $f_N \in \Phi_l$ and $z \in G$. For $x, y \in G$, $(f_N)_z(x,y) = f_N(xz, yz) = N(z^{-1}y^{-1}xz) = N_z(y^{-1}x) \in \Phi_l$. Thus, Φ_l is right G-closed (condition (D)). And conversely if Φ_l is right G-closed, then for $N \in \mathcal{N}$ and $z \in G$, $(f_N)_z \in \Phi_l$ implies some quasi-norm $M \in \mathcal{N}$ satisfies $N_z(x) = M(x)$, i.e. $N_z \in \mathcal{N}$.Therefore, $\mathcal{N}$ is conjugation-closed (condition (D')).

Finally, if $N(x) = N(x^{-1})$, then $f_N(x,e) = f_N(e,x)$. And by Theorem 2.5, the inversion map is continuous. Hence, all conditions are equivalent. □

Example 2.14. 1. Additive group $(\mathbb{R},+)$. Let Ṇ $= \{N(x) = |x|\}$. Then $\Phi = \{f_N(x,y) = |y-x|\}$, which is right-closed. The topology generated is the usual topology, making $(\mathbb{R},+)$ a topological group.

2. Matrix group $GL_n(\mathbb{R})$. Let $N(A) = \|A\|$ be an operator norm. For $z \in GL_n(\mathbb{R})$, $N_z(A) = \|z^{-1}Az\| \in$ Ṇ, so the family Ṇ is conjugation-closed. The induced topology is the usual norm topology, making $GL_n(\mathbb{R})$ a topological group.

3. Locally convex vector spaces $(V,+)$. Let Ṇ $= \{N_p(x) = p(x)\}$, p a seminorm on V. Since $(V,+)$ is abelian group, conjugation is trivial, and thus Ṇ is automatically conjugation-closed. The generated topology coincides with the usual locally convex topology, making $(V,+)$ a topological group.

4. Free groups F_n with word length. Let F_n be the free group on n generators, and let $N(w) = N(w^{-1})$ be the reduced word length. Conjugation-closure holds up to equivalence (the length changes by bounded length), and symmetry holds $N(w) = N(w^{-1})$. The topology generated by $f_N(x,y) = |y^{-1}x|$ is the standard word-metric topology on F_n making it a topological group.

Corollary 2.15. *If Ṇ is conjugation-closed, then the left topology generated by Ṇ makes G a topological group.*

Proof. This follows immediately from Theorem 2.13 ($(D') \Leftrightarrow (B')$). □

Theorem 2.16. *Let τ be a left topology on a group G for which the left translations are continuous. Then $(G,\cdot,\tau)$ is a topological group if and only if, for every neighborhood U of the identity e, there exists a continuous quasi-norm N on G such that $N(G\setminus U) = \{1\}$ and $N(U) \subseteq [0,1)$.*

Proof. $(\Rightarrow)$. Assume $(G,\cdot,\tau)$ is a topological group. let U be a neighborhood of e. Then there exists a symmetric neighborhood V of e such that $V\cdot V \subseteq U$. According to $[4, Theorem 1]$, there exists a quasi-norm N on G such that $N(G\setminus V) = \{1\}$ and $N(V) \subseteq [0,1)$. Since $V \subseteq U$, it follows that $G\setminus V \subseteq G\setminus U$. Therefore, for any $x \in G\setminus U$, $N(x) = 1$, i.e., $N(G\setminus U) = \{1\}$.

$(\Leftarrow)$. Assume that the condition holds. Since τ is a left topology for which the left translations are continuous, the multiplication is continuous (see [1]). It remains to prove that inversion is continuous.

Let U be an arbitrary neighborhood of e. By hypothesis, there exists a continuous quasi-norm N such that $N(G\setminus U) = \{1\}$ and $N(U) \subseteq [0,1)$. Define $W = N^{-1}([0,1))$. Since N is continuous and $N(e) = 0$, the set W is an open neighborhood of e. Moreover, if $x \in W$, then $N(x) < 1$, which implies $x \in U$. Therefore $W \subseteq U$. Now, consider the set U^{-1}. Since left translations are homeomorphisms and U is a neighborhood of e, U^{-1} is also neighborhood of e. By applying the hepothesis to U^{-1}, there exists a continuous quasi-norm M such that $M(G\setminus U^{-1}) = \{1\}$ and $M(U^{-1}) \subseteq [0,1)$. Define $V = M^{-1}([0,1))$. Since M is continuous and $M(e) = 0$, V is an open neighborhood of e. Furthermore, if $x \in V$, then $M(x) < 1$, which implies $x \in U^{-1}$. Therefore $V \subseteq U^{-1}$, which means

$V^{-1} \subseteq U$. This proves that inversion is continuous at e, and by left-invariance, it is continuous everywhere. Therefore, $(G, \cdot, \tau)$ is a topological group. □

Remark 2.17. The necessity of the continuity of inversion is highlighted by the Sorgenfrey line. Endowed with the usual addition and the Sorgenfrey topology, it is a paratopological group (multiplication is jointly continuous and translations are continuous) but inversion is not continuous. In this case the condition in Theorem 2.14 fails, there does not exist a continuous quasi-norm for every neighborhood of the identity. The Sorgenfrey line is an example that demonstrates the necessity of the continuity in this theorem and shows that the requirement of the continuity of inversion is essential and cannot be omitted.

Theorem 2.18. *Let N be a norm on the group G. Then, the left and right topologies coincide, $\tau_l = \tau_r$, and $(G, \cdot, \tau)$ is a topological group.*

Proof. Basic neighborhood of the identity e are given by

$V^l_\epsilon = \{x \in G : f_N(x, e) < \epsilon\} = \{x \in G : N(e^{-1}x) < \epsilon\} = \{x \in G : N(x) < \epsilon\}$.

$V^r_\epsilon = \{x \in G : g_N(x, e) < \epsilon\} = \{x \in G : N(xe^{-1}) < \epsilon\} = \{x \in G : N(x) < \epsilon\}$.

Thus, these neighborhoods coincide, and hence $\tau_l = \tau_r$.

Now, let $V = \{x \in G : N(x) < \epsilon\}$. Since N is a norm, $N(x) = N(x^{-1})$. Then $V^{-1} \subseteq V$, proving inversion is continuous at e. By translation invariance, inversion is continuous everywhere.

The multiplication is also continuous. Indeed, let $V = \{z \in G : N(z) < \epsilon\}$. Since N is a quasi-norm, there exists $K \geqslant 1$ with $N(xy) \leqslant K(N(x) + N(y))$. Choose $\delta = \frac{\epsilon}{2K}$ and $U = \{x \in G : N(x) < \delta\}$. Then for $x, y \in U$, $N(xy) \leqslant K(N(x) + N(y)) < K(2\delta) = \epsilon$. Thus, $xy \in V$, proving joint continuity at (e, e). Left invariance extends this to all points. Hence, $(G, \cdot, \tau)$ is a topological group. □

Example 2.19. 1. Additive group $(R, +)$. Let $N(x) = |x|$. Then $\tau_l = \tau_r$, is usual topology, making $(R, +)$ a topological group.

2. Matrix group $GL_n(R)$. Let $N(A) = \|A - I\|$ for an operator norm $\|\cdot\|$. Then $\tau_l = \tau_r$,coincides with the standard norm topology, making $GL_n(R)$ a topological group.

3. Locally convex vector spaces $(V, +)$. Let Ņ $= \{N_p(x) = p(x)\}$ be a seminorms defining on V a locally convex topology. Then $\tau_l = \tau_r$, making $(V, +)$ a topological group.

Theorem 2.20. *Let Ņ $= \{N_i\}_{i \in I}$ be a family of quasi-norms on a group G. Let τ_l be the left topology generated by the family of left-invariant quasi-pseudometrics $\{f_{N_i}(x, y) = N_i(y^{-1}x) : N_i \in$ Ņ$\}$, and let τ_r be the right topology generated by the family of right-invariant quasi-pseudometrics $\{g_{N_i}(x, y) = N_i(xy^{-1}) : N_i \in$ Ņ$\}$. Then $(G, \cdot, \tau_l, \tau_r)$ is a bitopological group if and only if both (G, τ_l) and (G, τ_r) are topological groups and the identity map $id : (G, \tau_l) \to (G, \tau_r)$ is a homeomorphism.*

Furthermore, a sufficient condition for these properties is that every quasi-norm $N_i \in$ Ņ is a norm.

Proof. (⇒). Assume that $(G, \cdot, \tau_l, \tau_r)$ is a bitopological group. Then the multiplication is continuous from $(G, \tau_l) \times (G, \tau_r)$ into both (G, τ_l) and (G, τ_r). And inversion is continuous in both (G, τ_l) and (G, τ_r). Hence $(G, \cdot, \tau_l)$, $(G, \cdot, \tau_r)$ are topological groups. Indeed, take $a \in G$. A basic τ_r-neighborhood of a is $V = \{y \in G : \underset{i\in J}{max} g_{N_i}(a, y) < \epsilon\} = \{y \in G : \underset{i\in J}{max} N_i(ay^{-1}) < \epsilon\}$, for some finite $J \subseteq I$, $\epsilon > 0$. By continuity of multiplication at (e, a) in the mixed topology $(G, \tau_l) \times (G, \tau_r) \to (G, \tau_r)$, there exist a τ_l-neighborhood U of e and a τ_r-neighborhood W of a such that $U \cdot W \subseteq V$. Since $a \in W$, $U \cdot a \subseteq U \cdot W \subseteq V$. Because right translations are homeomorphisms in τ_l, the set $U \cdot a$ is τ_l-open, hence V is τ_l-open. Thus $id : (G, \tau_l) \to (G, \tau_r)$ is continuous. So the multiplication in (G, τ_l) is continuous because it is the composition $(G, \tau_l) \times (G, \tau_l) \xrightarrow{id\times id} (G, \tau_l) \times (G, \tau_r) \xrightarrow{\mu} (G, \tau_l)$, with $\mu(x, y) = xy$. The first map is continuous since id is, and the second by assumption.

A symmetric argument shows that multiplication is continuous in (G, τ_r).

Now let us show that id is a homeomorphism. We already proved $id : (G, \tau_l) \to (G, \tau_r)$ is continuous. For the inverse, take $a \in G$. A basic τ_l-neighborhood of a is $U = \{y \in G : \underset{i\in J}{max} F_{N_i}(a, y) < \epsilon\} = \{y \in G : \underset{i\in J}{max} N_i(ay^{-1}) < \epsilon\}$. By continuity of multiplication at (a, e) in the mixed topology $(G, \tau_r) \times (G, \tau_l) \to (G, \tau_l)$, there exist a τ_r-neighborhood V of a and a τ_l-neighborhood W of e such that $V \cdot W \subseteq U$. Since $a \in V$, $a \cdot W \subseteq V \cdot W \subseteq U$. Because left translations are homeomorphisms in τ_r, the set $a \cdot W$ is τ_r-open, hence U is τ_r-open. Thus $id^{-1} : (G, \tau_r) \to (G, \tau_l)$ is continuous. So id is a homeomorphism.

(⇐). Conversely, assume that both $(G, \cdot, \tau_l)$, $(G, \cdot, \tau_r)$ are topological groups, and that $id : (G, \tau_l) \to (G, \tau_r)$ is a homeomorphism. Then inversion is continuous in both by assumption; composing the continuous map $(x, y) \mapsto (x, id^{-1}(y))$ is continuous from $(G, \tau_l) \times (G, \tau_r)$ to $(G, \tau_l) \times (G, \tau_l)$ with multiplication in (G, τ_l) shows continuity of the multiplication $(G, \tau_l) \times (G, \tau_r) \to (G, \tau_l)$; the map $(G, \tau_l) \times (G, \tau_r) \to (G, \tau_r)$ is continuous by symmetry. Hence $(G, \cdot, \tau_{f_N}, \tau_{g_N})$ is a bitopological group.

At the end, suppose each N_i is a norm, i.e., $N_i(x) = N_i(x^{-1})$. Then $f_{N_i}(x, y) = N_i(y^{-1}x) = N_i((y^{-1}x)^{-1}) = N_i(x^{-1}y)$ and $g_{N_i}(x, y) = N_i(xy^{-1}) = N_i((xy^{-1})^{-1}) = N_i(yx^{-1})$. In a neighborhood of e in τ_l has the form $\{x \in G : N_i(x) < \epsilon\}$, while in τ_r it is $\{x \in G : N_i(x^{-1}) < \epsilon\}$. By symmetry this coincide, so $\tau_l = \tau_r$. Thus, this common topology is generated by a family of norms, which, by Theorem 2.13 makes G a topological group. With $\tau_l = \tau_r$, both conditions are satisfied, hence $(G, \cdot, \tau_l, \tau_r)$ is a bitopological group. □

Theorem 2.21. *Let $\underset{\sim}{N} = \{N_i\}_{i\in I}$ be a family of quasi-norms on a group G. Let τ_l be the left topology generated by the family of left-invariant quasi-pseudometrics $\{f_{N_i}(x, y) = N_i(y^{-1}x) : N_i \in \underset{\sim}{N}\}$, and let τ_r be the right topology generated by the family of right-invariant quasi-pseudometrics $\{g_{N_i}(x, y) = N_i(xy^{-1}) : N_i \in \underset{\sim}{N}\}$. A sufficient condition for $(G, \cdot, \tau_l, \tau_r)$ to be a bitopological group is that $\underset{\sim}{N}$ satisfies the following asymmetric symmetry condition: there exists a constant $C \geqslant 1$ such* *that for every* $N_i \in \mathcal{N}$ *and every* $x \in G$, $N_i(x^{-1}) \leqslant CN_i(x)$.

If this condition holds, then both $(G, \cdot, \tau_l)$ *and* $(G, \cdot, \tau_r)$ *are topological groups. If in addition, the identity map* $id : (G, \tau_l) \to (G, \tau_r)$ *is a homeomorphism, then* $(G, \cdot, \tau_l, \tau_r)$ *is a bitopological group.*

Proof. We prove the claim for $(G, \cdot, \tau_l)$; the argument for $(G, \cdot, \tau_r)$ is analogous.

Since τ_l is generated by left-invariant quasi-pseudometrics, all left translations are homeomorphisms. Hence, to check continuity of group operations, it suffices to verify continuity at the identity e. Let $W = \{y \in G : N_i(y) < \epsilon, N_i \in \mathcal{N}, \epsilon > 0\}$ be a basic τ_l-neighborhood of e. By the asymmetric symmetry condition, $N_i(x^{-1}) \leqslant CN_i(x)$, define $U = \{x \in G : N_i(x) < \frac{\epsilon}{C}\}$. Then U is a τ_l-neighborhood of e. If $x \in U$, then $N_i(x^{-1}) \leqslant C \cdot N_i(x) < C \cdot \frac{\epsilon}{C} = \epsilon$, so $x^{-1} \in W$. Thus, $U^{-1} \subseteq W$. Hence inversion is continuous at e, and therefore everywhere.

Let us now show the joint continuity of multiplication in (e, e). Let $W = \{z \in G : N_i(z) < \epsilon, N_i \in \mathcal{N}, \epsilon > 0\}$ be a basic τ_l-neighborhood of e. Since each N_i is a quasi-norm, there exists a constant $K_i \geqslant 1$ such that $N_i(xy) \leqslant K_i(N_i(x) + N_i(y))$, for all $x, y \in G$. Choose $\delta = \frac{\epsilon}{2K_i}$, and define $U = \{x \in G : N_i(x) < \delta, N_i \in \mathcal{N}, \delta > 0\}$, then for $x, y \in U$, $N_i(xy) \leqslant K_i(N_i(x) + N_i(y)) < K_i(\delta + \delta) = 2K_i\delta = \epsilon$, so $xy \in W$. This proves that multiplication is jointly continuous at (e, e), and hence everywhere.

Therefore, $(G, \cdot, \tau_l)$ is a topological group. By a symmetric argument, $(G, \cdot, \tau_r)$ is also a topological group.

By Theorem 2.20, $(G, \cdot, \tau_l, \tau_r)$ is a bitopological group if and only if $(G, \cdot, \tau_l)$, $(G, \cdot, \tau_r)$ are topological groups, and the identity map $id : (G, \tau_l) \to (G, \tau_r)$ is a homeomorphism.

The asymmetric symmetry condition guarantees the first requirememt. If, in addition, the identity map is a homeomorphism, then the bitopological group structure follows. □

Corollary 2.22. *If every quasi-norm* $N_i \in \mathcal{N}$ *is, in fact a norm, then* $(G, \cdot, \tau_l, \tau_r)$ *is a bitopological group. In this case, the topologies* τ_l *and* τ_r *coincide.*

Proof. If each N_i is a norm, then The asymmetric symmetry condition is satisfied with constant $C = 1$, since $N_i(x^{-1}) = N_i(x)$. Therefore, by Theoerem 2.21, both $(G, \cdot, \tau_l)$, $(G, \cdot, \tau_r)$ are topological groups. Moreover, the neighborhood bases at the identity coincide : $B^l_{N_i}(e, \epsilon) = \{x \in G : N_i(x) < \epsilon\}$ and $B^r_{N_i}(e, \epsilon) = \{x \in G : N_i(x) < \epsilon\}$

Hence,$\tau_l = \tau_r$. Since both topologies are translation-invariant, it follows that the identity map $id : (G, \tau_l) \to (G, \tau_r)$ is trivially a homeomorphism. Therefore, by Theorem 2.20 $(G, \cdot, \tau_l, \tau_r)$ is a bitopological group. □

Note. Corollary 2.22 shows that if the generating quasi-norms are symmetric (i.e. true norms), then the natural topologies τ_l and τ_r automatically coincide. In this case, the bitopological structure does not yield new phenomena; it reduces to the familiar framework of a single topological group.

The real significance of Theorem 2.21 emerges in the asymmetric case, where quasi-norms are not symmetric and the left and right topologies may genuinely differ. The asymmetric symmetry condition then ensures that both topologies are still compatible enough to form a bitopological group.

Example 2.23. 1. In the additive group $(\mathbb{R},+)$ with the norm $N(x)=|x|$, the left- and right-invariant quasi-pseudometrics are $f_N(x,y)=|x-y|$, $g_N(x,y)=|x-y|$. Hence, $\tau_l=\tau_r$ is the standard Euclidean topology. Thus by Corollary 2.22 $(\mathbb{R},+,\tau_l,\tau_r)$ is a bitopological group, which coincides with the usual topological group $(\mathbb{R},+,\tau_l)$.

2. Let $G=(\mathbb{R}^2,+)$ be the additive group of the plane. Define the quasi-norm $N(x,y)=|x|+2|y|$. This is symmetric, since $N((x,y)^{-1})=N(-x,-y)=|x|+2|y|=N(x,y)$. Thus, the asymmetric symmetry condition of Theorem 2.21 is trivially satisfied with constant $C=1$.The induced left and right quasi-pseudometrics coincide. For $(x,y),(u,v)\in G$, $f_N((x,y),(u,v))=N((u,v)^{-1}+(x,y))=N(x-u,y-v)=|x-u|+2|y-v|$, $g_N((x,y),(u,v))=N((x,y)+(u,v)^{-1})=N(x-u,y-v)=|x-u|+2|y-v|$. Hence $f_N=g_N$, so the generated topologies coincide:$\tau_l=\tau_r$, which is precisely the topology induced by the norm $\|(x,y)\|=|x|+2|y|$. Therefore, Theorem 2.21 ensures that both $(G,+,\tau_l)$, $(G,+,\tau_r)$ are topological groups. However, the resulting bitopological structure $(\mathbb{R}^2,+,\tau_l,\tau_r)$ is trivial, since the two topologies coincide.

Proposition 2.24. *Let $(G,\cdot,\tau_l,\tau_r)$ be a bitopological group generated by a family of quasi-norms $\mathcal{N}=\{N_i\}_{i\in I}$. Then the following are equivalent:*

1. *The left topology τ_l is T_1.*
2. *The right topology τ_r is T_1.*
3. *The family $\mathcal{N}$ is separating; i.e., for every $g\neq e$ in G, there exists some $N_i\in\mathcal{N}$ such that $N_i(g)>0$.*

In particular, if $\mathcal{N}$ is separating, then the bitopological group $(G,\cdot,\tau_l,\tau_r)$ satisfies the T_1 separation axiom.

Proof. In any topological group (and thus in both τ_l and τ_r), the T_1 property is equivalent to the singleton $\{e\}$ being closed. This holds if and only if the intersection of all neighborhoods of e is $\{e\}$. We define the neighborhood bases at the identity e. The left neighborhood base at e, $B_l(e)$, consists of all finite intersections of sets of the form $U_{i,\epsilon}=\{h\in G: N_i(h)<\epsilon, N_i\in\mathcal{N},\epsilon>0\}$. Similarly, and the right neighborhood base at e, $B_r(e)$, consists of all finite intersections of sets of $V_{i,\epsilon}=\{h\in G: N_i(h^{-1})<\epsilon, N_i\in\mathcal{N},\epsilon>0\}$.

Thus, the T_1 condition is equivalent to $\cap B_l(e)=\{e\}$ and $\cap B_r(e)=\{e\}$.

(3) $\Rightarrow$ (1): Assume $\mathcal{N}$ is separating. Let $g\neq e$, Then there exists $N_i\in\mathcal{N}$ such that $N_i(g)=\delta>0$. Set $\epsilon=\frac{\delta}{2}$, then basic neighborhood $U_{i,\epsilon}$ contains e but $g\notin U_{i,\epsilon}$. Hence $\cap B_l(e)=\{e\}$, and thus τ_l is T_1.

(1) ⇒ (3): Assume τ_l is T_1. Let $g \neq e$. Then there exists a neighborhood $U \in B_l(e)$ such that $g \notin U$. Write $U = \cap_{k=1}^{m} U_{i_k,\epsilon_k}$. Then $g \neq e$ implies that for some k, $g \notin U_{i_k,\epsilon_k}$, hence $N_{i_k}(g) \geqslant \epsilon_k > 0$. Therefore, Ņ is separating.

(3) ⇒ (2): Assume Ņ is separating. Let $g \neq e$. Then $g^{-1} \neq e$, so there exists $N_j \in$ Ņ such that $N_j(g^{-1}) = \delta > 0$. Set $\epsilon = \frac{\delta}{2}$. Then $g \notin V_{j,\epsilon}$. Thus, $g \notin \cap B_r(e)$, so τ_r is T_1.

(2) ⇒ (3): Assume τ_r is T_1. Let $g \neq e$. Then there exists a neighborhood $V = \cap_{k=1}^{m} V_{i_k,\epsilon_k} \in B_r(e)$ such that $g \notin V$. Hence, for some k, $g \notin V_{i_k,\epsilon_k}$, so $N_{i_k}(g^{-1}) \geqslant \epsilon_k > 0$. Let $h = g^{-1}$. Then $h \neq e$ and $N_{i_k}(h) \geqslant> 0$, so Ņ is separating.
Therefore the three statements are equivalent.

By definition, bitopological group is T_1 iff both τ_l and τ_r are T_1. Since (1) ⇔ (2) ⇔ (3), this occurs precisely when Ņ is separating. □

Remark 2.25. *By translation invariance of τ_l and τ_r, if $(G, \cdot, \tau_l, \tau_r)$ is T_1, then every singleton $\{g\}$, $g \in G$, is closed. Indeed, for any $g \in G$, the left translation $\lambda_g : h \mapsto gh$ is a homeomorphism of (G, τ_l) onto itself, and similarly, $\rho_g : h \mapsto hg$ is a homeomorphism of (G, τ_r) onto itself. Hence, the closedness of $\{e\}$ implies the closedness of $\{g\}$ in both topologies.*

As an example of a bitopological group satisfying the T_1 property, consider the additive group $G = (R^2; +)$ equiped with the quasi-norm $N(x, y) = |x| + 2|y|$.

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