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Unbounded direct sums of cyclic p-groups are characteristically inert socle-regular groups

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Abstract. It was proved by Chekhlov-Danchev in Commun. Algebra (2022) that unbounded direct sums of cyclic p-groups are (weakly) characteristically inert socle-regular for any prime p. This short note aims to give a new more direct and transparent proof of this fact, provided the prime number p is odd, that is, different to 2.

1. Introduction and motivation

Throughout the present work, the term *group* will mean an *additively written Abelian group*. Our notation and terminology are at all standard and may be found in the books [5, 6, 7]. For instance, the term "summand" will, everywhere, stand for "direct summand".

Let us remember the notion that two subgroups B and C of a group G are called *commensurable*, and write for short that $B \sim C$, provided that both quotients $(B+C)/B$ and $(B+C)/C$ are finite. Note that this is obviously a symmetric relation, and thus $C \sim B$ holds.

Referring to [1], [2], [3] for a more account, we shall say that a group G is *fully inert socle-regular* (respectively, *characteristically inert socle-regular*) if, for all infinite fully inert (respectively, infinite characteristically inert) subgroups H of G, there is an ordinal α, depending on H, such that $H[p] \sim (p^\alpha G)[p]$.

Alternatively, a group G is said to be *weakly fully inert socle-regular* (respectively, *weakly characteristically socle-regular*) if, for all infinite fully inert (respectively, infinite characteristically inert) subgroups H of G, there is an ordinal α, depending on H, such that $p^\alpha G \neq \{0\}$ and $H[p] \cap (p^\alpha G)[p]$ is of finite index in $(p^\alpha G)[p]$.

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It was shown in [2, Corollary 1.5] that direct sums of cyclic p-groups are (weakly) fully inert socle-regular for all primes p. However, for the unbounded case, this assertion was improved in [1, Proposition 3.3 (1)] by showing that unbounded direct sums of cyclic p-groups are weakly characteristically inert socle-regular. This can be even refined with the aid of [1, Theorem 3.1] establishing that unbounded direct sums of cyclic p-groups are, actually, characteristically inert socle-regular, because for separable p-groups the concepts of weak characteristically inert socle-regularity and characteristically inert socle-regularity do coincide always.

So, stimulated by these results, in this brief note we want to look at the possibility to ensure a new more plain and less conceptual proof in the (weak) case of characteristically inert socle-regularity. We, however, succeed to do that only in the case where p is an odd prime.

2. The main result

The following statement was established in [3] and is our key instrument to prove the major result below.

Proposition 2.1. *Suppose $p \neq 2$ is a prime. If $G = \bigoplus_{n\geqslant 1} B_n$ is an unbounded direct sum of cyclic groups, where each B_n is either cyclic of order p^n or zero, then G is (weakly) characteristically inert socle-regular.*

Our chief achievement here is the following. Notice that, for convenience of our further writing, we shall hereafter write $p^\alpha G[p]$ instead of $(p^\alpha G)[p]$, for any ordinal α.

Theorem 2.2. *Suppose $p > 2$ is a prime. Then, any unbounded direct sum of cyclic p-groups is (weakly) characteristically inert socle-regular.*

Proof. As a first step we take each homocyclic component of G and, if it has odd rank we split off a single cyclic summand. Gathering these together we can re-write $G = F \oplus E$, where E is a direct sum of cyclic groups such that each homocyclic component is either finite of even rank or else it has infinite; the summand F is a direct sum of cyclic groups where the homocyclic components are of rank 1 or zero. Let I_0 denote the number of non-zero homocyclic components in E and I_1 the number in F.

Let C be an arbitrary infinite characteristically inert subgroup of G; if $E = 0$, then employing Proposition 2.1 we have that $(C \cap p^n F)[p]$ is commensurable with $p^nF[p]$ for some integer n and it follows immediately that

$$(C \cap p^nG)[p] \sim p^nG[p].$$

So, we may assume henceforth that $E \neq 0$. Notice that E has the property that it can be written in the form $E = X \oplus X$ for some non-zero group X and hence, as is well-known, each endomorphism of E is a sum of at most 3 automorphisms of E, and also the identity map 1_F can be expressed as a sum of two automorphisms of F.

We are now in a position to utilize [3, Proposition 2.7 (i),(ii)] so that

$$C \sim (C \cap E) \oplus (C \cap F)$$

and $C \cap E, C \cap F$ are characteristically inert in E, F, respectively. Note also that I_0, I_1 cannot both be finite since this would contradict G being unbounded. We discuss the resulting three possibilities separately as follows:

CASE 1: I_1 is finite.

It follows at once that F is finite, so that $p^nG[p] \sim p^nE[p]$ for all n. Applying [3, Proposition 4.2] to E and its characteristically inert subgroup $C \cap E$, we see that $C \cap E$ is actually fully inert in E, and the latter is a direct sum of cyclic groups which is known to be fully inert socle-regular (by, for example, [2, Corollary 1.5]). Furthermore, since

$$C \sim (C \cap E) \oplus (C \cap F)$$

and the last summand is finite, we have that $C \cap E$ is infinite. Thus, there exists an integer m such that

$$(C \cap E)[p] \sim p^mE[p] \sim p^mG[p],$$

and hence in this case we even have that $C[p] \sim p^mG[p]$.

CASE 2: I_0 is finite.

In this case I_1 is infinite and E is bounded while F is unbounded and, as before, we have that

$$C \sim (C \cap E) \oplus (C \cap F).$$

Assume we have shown that $C \cap F$ *is infinite.* Then, $(C \cap F)[p]$ is, simultaneously infinite and characteristically inert in F, a direct sum of cyclic groups, where each homocyclic component is either zero or of rank 1. Next, it follows from Proposition 2.1 that there is an integer R such that

$$(C \cap p^R F)[p] \sim p^R F[p];$$

in fact, it follows by standard properties of subgroups of finite index (see, for instance, Section 2 in [2]) that

$$(C \cap p^t F)[p] \sim p^t F[p]$$

for all $t \geqslant R$.

Furthermore, since E is bounded, one sees that $p^N G = p^N F$ for some integer N, so that $p^k G = p^k F$ for all $k \geqslant N$ and thus

$$(C \cap p^k F)[p] = (C \cap p^k G)[p]$$

for all $k \geqslant N$. Choose $m = \max\{R, N\}$, and then it follows that

$$(C \cap p^m G)[p] = (C \cap p^m F)[p] \sim p^m F[p] = p^m G[p].$$

So, to complete Case 2, we need to show that $C \cap F$ is infinite. To that goal, assume for a contradiction that $C \cap F$ is finite. Note firstly that, we may then assume, since

$$C \sim (C \cap E) \oplus (C \cap F),$$

that $C \cap E$ is infinite. Now, $(C \cap E)[p]$ is then an infinite subsocle of the direct sum of cyclic groups E, so there is a pure subgroup, L say, of E with the property $L[p] = (C \cap E)[p]$. However, since E is bounded here, L is a summand of E and also is an infinite direct sum of cyclic groups. But, it is well known (see, for example, [4, Lemma 3.2]) that we can construct a homomorphism $f : L \to F$, and hence by trivial extension a mapping $E \to F$, such that $f(L[p])$ is infinite. Now, apply [3, Proposition 2.7 (iii)] to this situation to get that

$$C \sim C + f(C \cap E).$$

Therefore, we have that

$$f(C \cap E)/(C \cap f(C \cap E))$$

is finite. But the denominator in the last expression is a subgroup of $C \cap F$ and so it must be finite by our assumption, forcing

$$f((C \cap E)[p]) = f(L[p])$$

to be finite - contradiction. Thus, $C \cap F$ is infinite and the proof in Case 2 is complete after all, as wanted.

CASE 3: Both I_0 and I_1 are infinite.

In this case, both E and F are unbounded and, as before, we have that $C \cap E, C \cap F$ are characteristically inert in E, F, respectively. Note also that 1_E is again a sum of two automorphisms of E as well as the endomorphisms of E are sums of at most 3 automorphisms of E, and so by [3, Proposition 2.7(ii)]

$$C \sim (C \cap E) \oplus (C \cap F)$$

with $C \cap E, C \cap F$ characteristically inert in E, F, respectively.

If $C \cap E, C \cap F$ are both infinite, and as we observed above that the endomorphisms of E are sums of at most 3 automorphisms of E, we have with [3, Proposition 4.2] at hand that $C \cap E$ is actually fully inert in E, so there is an integer, N say, with

$$(E \cap C) \cap p^N E[p] \sim p^N E[p].$$

We also have that $C \cap F$ is characteristically inert in F and so Proposition 2.1 allows us to derive that, there is an integer R such that

$$(C \cap F) \cap p^R F[p] \sim p^R F[p].$$

Then, arguing as in Case 2, if we take $m = \max\{N, R\}$, we can get that

$$(C \cap p^m E[p]) \oplus (C \cap p^m F[p]) \sim p^m E[p] \oplus p^m F[p] = p^m G[p].$$

However, as $C \cap p^m G[p]$ contains the first term in the last expression, we deduce

$$C \cap p^m G[p] \sim p^m G[p],$$

as desired.

The question then arises can either of the intersections $C \cap E, C \cap F$ be finite. To answer this, we make use of the following simple result.

Lemma 2.3. *Suppose that H is an infinite characteristically inert subgroup of a group $A = L \oplus M$ and $H \sim (H \cap L) \oplus (H \cap M)$. Then, if $H \cap L$ is finite and $f : M \to L$ is an arbitrary homomorphism, the image $f(H \cap M)$ is also finite.*

Proof. Exploiting [3, Proposition 2.7(iii)], we obtain that

$$H \sim H + f(H \cap M),$$

so that the quotient

$$f(H \cap M)/(H \cap f(H \cap M))$$

is finite. But, the denominator of the last term is contained in the finite group $H \cap L$, so that $f(H \cap M)$ is also finite, as required. □

Returning to the argument in Case 3, consider firstly the possibility that $C \cap F$ is finite. In this situation, we must have that $C \cap E$ is infinite and, as saw above, that the endomorphisms of E are sums of at most 3 automorphisms of E. It follows now with the aid of [3, Proposition 4.2] that $C \cap E$ is actually fully inert in E and, there is an integer n, such that

$$(C \cap E)[p] \sim p^n E[p].$$

As we are in Case 3, E, F are both unbounded and so it follows - see, for e.g., the proof of [4, Lemma 3.2] - that there is an unbounded summand, Y say, of E and a homomorphism $\sigma : E \to F$ such that $\sigma \restriction Y[p]$ is an injection. Since Y is unbounded, $p^n Y[p]$ is infinite and contained in $p^n E[p]$, so that $\sigma(p^n E[p])$ is infinite. And since

$$(C \cap E)[p] \sim p^n E[p],$$

we then would have that $\sigma((C \cap E)[p])$ is infinite, contrary to Lemma 2.3. Consequently, in this case, $C \cap F$ cannot be finite, as pursued.

The argument to show that $C \cap E$ cannot be finite is broadly similar. In fact, if $C \cap E$ is finite, then we must have that $C \cap F$ is infinite and characteristically inert in F. It then follows from Proposition 2.1 that there is an integer n with

$$(C \cap F)[p] \cap p^n F \sim p^n F[p],$$

i.e.,

$$(C \cap p^n F)[p] \sim p^n F[p].$$

As we argued in Case 3, F is unbounded, so that $p^nF[p]$ is infinite and the last commensurability relation gives us that $(C \cap p^nF)[p]$ is also infinite. Now, we repeat the argument in the previous paragraph and thus we are again led to a contradiction. So, $C \cap E$ cannot be finite, as expected.

The theorem is proved after all. $\square$

It is worthwhile noticing that our theorem will be true for an arbitrary prime if Proposition 2.1 remains valid for the *standard* 2-group G. Recall that, in the current situation, we know that the automorphism group of G does *not* additively generate the full endomorphism group of G and that the identity endomorphism 1_G is *not* the sum of two automorphisms of G, so that we do not immediately get a commensurability relation as in [3, Proposition 2.7]. Likewise, [3, Theorem 4.4] asserts that a totally projective p-group for any $p \neq 2$ is characteristically inert socle-regular, but however our proof is rather more easy and non-conceptual.

Also, in parallel to [1, Theorem 3.1], one can ask the challenging question: does it follow that separable weakly fully inert socle-regular p-groups are fully inert socle-regular?

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References

[1] **A.R. Chekhlov and P.V. Danchev**, *Weakly fully and characteristically inert socle-regular Abelian p-groups*, Commun. Algebra **50** (2022), no. 11, 4975 – 4987.

[2] **A.R. Chekhlov, P.V. Danchev and B. Goldsmith**, *On the socles of fully inert subgroups of Abelian p-groups*, Mediterr. J. Math. **18** (2021), no. 3, 122 – 142.

[3] **A.R. Chekhlov, P.V. Danchev and B. Goldsmith**, *On the socles of characteristically inert subgroups of Abelian p-groups*, Forum Math. **33** (2021), no. 4, 889 – 898.

[4] **B. Goldsmith and L. Salce**, *Fully inert subgroups of torsion-complete p-groups*, J. Algebra **555** (2020), 406 – 424.

[5] **L. Fuchs**, *Infinite Abelian Groups*, Vol. **I**, Acad. Press, New York and London, 1970.

[6] **L. Fuchs**, *Infinite Abelian Groups*, Vol. **II**, Acad. Press, New York and London, 1973.

[7] **L. Fuchs**, *Abelian Groups*, Springer, Switzerland 2015.

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