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Extensions of the Klein-4 group and signed products

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Abstract. Signed superquasigroups allow the creation of a (signed) super product for quasigroups similar to that which exists for superalgebras. They also permit the creation of the *quatedral loop*, a nonassociative loop with the same character table as the groups D_4 and Q_8. However, this loop was not created by directly taking the signed superproduct, but required some additional work. We show that it is impossible to create the quatedral loop as a signed superproduct of signed superquasigroups. In fact, all signed superproducts of order 4 signed superquasigroups are isotopic to Q_8 or D_4. A related result on signed superproducts applies to signed superproducts of an entire class of signed superquasigroups called *standard* superquasigroups: signed superquasigroups which are extensions of superloops by $\mathbb{Z}/_2$. Signed superproducts of such loops have particularly nice properties. A particularly interesting example of standard loops are hedral loops, which are extensions of $\mathbb{Z}/_2 \times \mathbb{Z}/_2$ by $\mathbb{Z}/_2$. The character tables of such loops are examined, and shown to share the character table of Q_8 or that of a non-cyclic abelian group of order 8. Finally, results on hedral loops are used to show that the 16 element octonion loop is not isotopic to a signed product of standard signed superquasigroups.

1. Introduction

Supersymmetry is especially important in the study of Clifford algebras and groups related to them. However, superstructure for non-associative objects is understudied. In [5], *signed* superquasigroups are introduced, as extensions of ordinary superquasigroups. Note, however, that the absence of an identity in general quasigroups requires a somewhat different definition of extension. Over piques, the ordinary definition in terms of short exact sequences works, and signed superquasigroups are extensions by $\mathbb{Z}/_2$.

For instance, $\mathbb{Z}/_2$ is a supergroup. It can be extended into a signed supergroup called the cyclic basic (signed) supergroup or C_4 (which as a group is just $\mathbb{Z}/_4$) or into the Boolean basic supergroup, B_4, which as a group is just $\mathbb{Z}/_2 \times \mathbb{Z}/_2$. These new quasigroups introduce negative and positive signs for each element of the quasigroup being extended. Signs satisfy $(-x)y = x(-y) = -(xy)$ and $(-x)(-y) = xy$ [5, §4].

This means that it is only necessary to consider multiplication on positive elements, which make up what is called the *transversal* of the quasigroup [5, Def. 4.1]. This

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allows the authors to extend the idea of a supertensor product on a superalgebra to superquasigroups. Recall that the supertensor product of two algebras is defined on the same set as the ordinary tensor product, but with product $(-1)^{|y|\,|z|}(x \otimes y)(z \otimes w)$, where absolute value gives the parity of the element, and is written $\widehat{\otimes}$. The supertensor product in algebras is especially useful because Clifford algebras satisfy $\mathrm{Cl}(V \oplus W) = \mathrm{Cl}(V)\widehat{\otimes}\mathrm{Cl}(W)$ [6, Thm. III.3.10]. Using the supertensor product of algebras as a model, the signed (super) product of signed superquasigroups is defined. The signed supertensor product of Boolean basic and cyclic basic supergroups, produce the "unit group" of Clifford algebras understood in the sense of [5, pg. 2-3]

An obvious first question is whether using superquasigroups of order 4 in signed superproducts other than the two nontrivially graded supergroups give any new and interesting structures. In [5], signed superproducts of other superquasigroups are examined. Particular interest is shown to two quasigroups isotopic to the cyclic basic signed supergroup C_4, labeled the left and right basic superquasigroups [5, §4.5]. In this paper we consider an additional 4 signed superquasigroups, which exhausts all order 4 signed superquasigroups up to isomorphism. We call these signed superquasigroups basic. The product of the right basic superquasigroup and C_4 in particular ultimately yields an interesting loop with properties similar to D_4 and Q_8, which the authors of the previously mentioned paper label the quatedral loop [5, §4]. The problem here is that after the taking of the product of the two signed superquasigroups, it is necessary to apply what the authors label a *modification* [5, Def. 4.7], which flips the sign of the multiplication of one pair of elements in the transversal of the multiplication table. Without such a modification, we show any such signed product of order 4 signed superquasigroups will yield a quasigroup isotopic to Q_8 or D_4. In fact, the result is more extensive than this.

If L is a loop, and L' is an extension of that loop by $\mathbb{Z}/_2$ we call L' a *standard signed superquasigroup*. If L' and M' are such standard signed superquasigroups, then there is a principal isotopy (such an isotopy exists for any signed superquasigroup); called the *signed isotopy* from L' (respectively, M') to a loop L'_S (M'_S) and an isotopy from $L'\widehat{\times}M'$ to a loop $(L'\widehat{\times}M')_S$, and $L'_S\widehat{\times}M'_S = (L'\widehat{\times}M')_S$ (Theorem 5.7). That is, the signed superproduct preserves a certain principal isotopy, which is referred to as the signed isotopy, since it only changes the signs of elements (Theorem 5.7). What this means is that, up to isotopy, the signed product of standard superquasigroups is just the signed superproducts of signed superloops.

Then, specific instances of loop extensions by $\mathbb{Z}/_2$ are examined; those which are extensions of $\mathbb{Z}/_2 \times \mathbb{Z}/_2$, or the Klein-4 group. These loops are called hedral loops, after the dihedral and quatedral loops of order 8, which are two prominent examples, along with the quaternion group. Conditions on the character tables of hedral loops are shown. It turns out that there are only three possible hedral loop character tables, all of which are character tables of at least one group. Finally, it is shown that the loop O_{16}, the smallest spanning loop of the real octonion algebra, is not isotopic to a signed product of signed superloops (Corollary 7.5), nor is it isotopic to a signed product of standard signed superquasigroups (Corollary 7.6).

Plan of the paper. In Section 2, some basic theorems and definitions from quasigroup theory are given. Most of these derive from [10]. Then an outline of some of the results and definitions from [5] are given in Section 3. First, superquasigroups are defined, then signed superquasigroups (Section 3.1). A list of all basic signed superquasigroups is then given in Section 3.2. Next, in Section 3.2, the main definition from [5] is recapitulated: namely, that of signed superproducts for signed superquasigroups. A result on the products of opposite signed superquasigroups is shown in Proposition 3.15. Some categorical results are also shown in Section 3.2. In Section §**??**, the products of what are called *basic* signed superquasigroup are examined and related to Clifford algebras.

In Section § standard superquasigroups are defined and it is shown in Theorem 5.7 that signed superproducts of standard signed superquasigroups are isotopic to products of certain signed superloops (Theorem 5.7). Of particular interest are extensions of the Klein-4 group, including D_4, Q_8, the Quatedral loop, and several other loops. These are called hedral loops after the dihedral group of of order 8. Some properties of such loops are shown in Section , including their possible character tables. Finally, in Section §, it is shown that the octonion loop O_{16} is not the signed product of signed superloops, nor of standard quasigroups.

Algebraic notation is often used for functions, so that composition is written fg instead of $g \circ f$. The expression $g \circ f(x)$ will usually be written x^{fg}.

2. Quasigroups and loops

Definition 2.1.

1. Combinatorially, a *quasigroup* is defined as a set Q with a binary operation $\cdot$ such that given a and b in Q there exists unique c and d in Q such that $a \cdot c = b$, $d \cdot a = b$.
2. Algebraically, a quasigroup is a set with three binary operations $\cdot$ (multiplication), $/$ (right division) and $\backslash$ (left division) such that the following axioms hold for all a and b:
   (a) $(a \cdot b)/b = a$
   (b) $b\backslash(b \cdot a) = a$
   (c) $(a/b) \cdot b = a$
   (d) $b \cdot (b\backslash a) = a$

A *loop* is a quasigroup with an identity. That is, an element e such that $e \circ g = g \circ e = g$.

A weaker distinguished element than an identity is a pointed idempotent, which is also often referred to as e. A *pique* has a pointed idempotent element. Loops are piques where the pointed idempotent is also an identity. The *opposite quasigroup* of Q is just the quasigroup on the set Q with opposite multiplication.

Every quasigroup is associated with a unique group called its *multiplication group*. Since multiplication on the left and right by a fixed quasigroup element x produces bijections (labeled L_x and R_x respectively) on the set of quasigroup elements, such bijections are elements of the permutation group on the set of quasigroup elements. The multiplication group of the quasigroup is the permutation group generated by L_x and R_x for all x in the quasigroup.

A *homotopy* from a quasigroup $(P, *)$ to a quasigroup $(Q, \cdot)$ is a set of three maps f, g and h such that $a^f \cdot b^g = (a * b)^h$. If these maps are bijections then the homotopy is an *isotopy*, and this isotopy is *principal* if h is the identity. All isotopies can be factored into an isomorphism and a principle isotopy Any principal isotopy defines a new multiplication given by $a * b = a^f \cdot b^g$. That this is always a quasigroup multiplication follows from the bijectivity of f and g.

Theorem 2.2. [1] *All quasigroups are isotopic to a loop.*

Proof. In fact any $x \in Q$ can become the identity under a principal isotopy. Find a and b such that $x = a \cdot b$ and define the maps $f = R_b^{-1}$ and $g = L_a^{-1}$. This defines a new multiplication on Q such that $c^f \cdot x^g = cR_b^{-1} \cdot bL_a L_a^{-1} = c$ and similar reasoning gives that $x^f \cdot c^g = c$. □

Lemma 2.3. *Any loop isotopic to a group is is isomorphic to that group.*

Definition 2.4. The product of two quasigroups $(Q, *)$ and $(P, \circ)$; $(Q \times P, * \times \circ)$ is the quasigroup defined on $Q \times P$ by $(q,p)(* \times \circ)(q',p') = (q * q', p \circ p')$.

Theorem 2.5 (First Isomorphism Theorem)**.** *The set of equivalences of a congruence $\sim$ of $(Q, *)$ forms a quasigroup $(Q/\sim, \cdot)$ and there is a standard projection π from Q to $Q/\sim$. There is an isomorphism between the image Q^π and $Q/\sim$.*

In a loop, a congruence can be thought of as a quotient of Q by the equivalency class of the identity element, which forms a subloop. That the kernel is indeed a subloop is clear, since if x and y in $(Q, *)$ are both sent to the identity element e by the projection $\pi : Q \to Q/\ker(\pi)$, then $\pi(x * y) = \pi(x) \cdot \pi(y) = e \cdot e = e$ while $\pi(x/y) = \pi(x)/\pi(y) = e/e = e$ and similarly for $x\backslash y$. Then the elements of $Q/\ker(\pi)$ can be viewed as cosets of the form $x \ker(\pi)$, as in groups.

Definition 2.6. *A* normal subloop *is a subloop which forms an equivalency class in a loop congruence.*

Lemma 2.7. [8] *If N is a subloop of a loop L with index 2, it is normal.*

3. Superquasigroups

Definition 3.1. [5] A *superquasigroup* is a quasigroup with a quasigroup map into $\mathbb{Z}/_2$ in which the kernel of this map is the *even parity* part and the rest of the group is the *odd parity* part. The symbol $|x|$ maps x to 0 if it is even and 1 if it is odd.

In particular, this means that the even part is a subquasigroup, because $|\cdot|$ is a quasigroup homomorphism, so the even part of a superquasigroup is a the pre-image of a subquasigroup, with multiplication and left and right division corresponding to addition modulo 2 in the image.

Definition 3.2. A superquasigroup homomorphism/homotopy is a quasigroup homomorphism/homotopy which preserves parity. These are also referred to as superhomomorphisms/superhomotopies.

Superquasigroups form a category with arrows given by superhomomorphisms. They also form a category when superhomomorphisms are replaced with superhomotopies.

Lemma 3.3. [5] *A superquasigroup with non-trivial odd part has odd and even parts of the same size.*

3.1. Signed (super)quasigroups

Definition 3.4. A *signed quasigroup* Q is a quasigroup such that there is a surjective map from Q to a quasigroup P with index 2.

If P is a pique, then this results in an exact sequence:

$$\{1\} \to \{\pm 1\} \to Q \to P \to \{1\}. \tag{3.1}$$

In this paper, we concern ourselves mostly with signed piques. A signed pique is an extension of the original pique by $\mathbb{Z}/_2$, with extension understood in the common sense as a short exact sequence. In a signed pique, the image of 1 in $\{\pm 1\}$ is represented in Q as 1 and that of -1 is represented as -1, regardless of whether the resulting quasigroup is a loop.

More generally, note that every equivalency class in the congruence mapping Q onto P has two elements. If the image of this congruence class under the projection from Q to P is x, call these elements x and $-x$. In this way, the subset of *positive* elements in Q is defined, taking one element from the pre-image in Q of each $x \in P$. Note that this division is arbitrary, and $-x$ could have been represented as x. The product $(-x) * (-y)$ is not equal to $(-x) * y$ or $x * (-y)$ since if $(-x) * (-y) = x * (-y)$, then $x = -x$. The same reasoning holds for $x * y$, so $x * y \neq (-x) * y, x * (-y)$. However, since all of these products have the same image in P as $x * y$, they are all in the same coset and there are only two elements in this coset by definition. Therefore, $(-x) * y = x * (-y)$ and $x * y = (-x) * (-y)$. Thus negative signs commute and the multiplication table on the subset of positive elements is sufficient to fully determine the multiplication table of the whole quasigroup [5].

Definition 3.5. [5] The transversal of a signed quasigroup is the set of positive elements.

Another way to construct a signed quasigroup is to take the multiplication table of a quasigroup P (from Definition 3.4) and assign negative signs to elements in the multiplication table. This gives the multiplication table of the transversal, which is sufficient to determine the whole multiplication table. Any such multiplication table will be that of a signed quasigroup as defined in Definition 3.4, since for any $p, p' \in P$, there exists $t, t' \in P$ such that $pt = p'$, $t'p = p'$ and thus either t or $-t$ in Q uniquely satisfies $p * x = p'$, for $p, p' \in Q$. The other satisfies the equation $p * (-x) = -p'$. The same reasoning holds for $t'p = p'$.

Definition 3.6. A signed quasigroup Q has set of positive elements Q^+ and set of negative elements Q^-.

Definition 3.7. A *signed superquasigroup* is a signed quasigroup which is also a superquasigroup, and such that x and $-x$ always have the same polarity.

Definition 3.8. A *standard* signed superquasigroup adds the additional condition that P is a loop.

3.1.1. Basic signed superquasigroups

Definition 3.9. A basic signed superquasigroup is a signed superquasigroup of order 2 or 4. We write the element sets of these signed superquasigroups $E \uplus O$, where E consists of even parity elements and O of odd parity elements.

(a) The *unit basic* super(quasi)group $C_2 = \{\pm 1\} \uplus \varnothing$ is the signed supergroup with multiplication table

	1
1	1

on its transversal $\{1\} \uplus \varnothing$ [5, §4.5].

(b) We now enumerate all order four signed superquasigroups such that $1^2 = 1$. That is, superquasigroups which are also piques with specified idempotent element 1. Each of these is isotopic to an order 4 signed supergroup.

The *Boolean basic* super(quasi)group $B_4 = \{\pm 1\} \uplus \{\pm i\}$ is the signed supergroup with multiplication table

	1	i
1	1	i
i	i	1

on the transversal $\{1\} \uplus \{i\}$ [5, §4.5].

(c) The *cyclic basic* super(quasi)group $C_4 = \{\pm 1\} \uplus \{\pm i\}$ is the signed supergroup with multiplication table

	1	i
1	1	i
i	i	-1

on the transversal $\{1\} \uplus \{i\}$ [5, §4.5].

(d) The *right basic* superquasigroup $C_4^{/} = \{\pm 1\} \uplus \{\pm i\}$ is the signed superquasigroup with multiplication table

	1	i
1	1	$-i$
i	i	1

on the transversal $\{1\} \uplus \{i\}$ [5, §4.5].

(e) The *left basic* superquasigroup $C_4^{\backslash} = \{\pm 1\} \uplus \{\pm i\}$ is the signed superquasigroup with multiplication table

	1	i
1	1	i
i	$-i$	1

on the transversal $\{1\} \uplus \{i\}$.

The right and left basic superquasigroups are described as being *chiral*, or having a handedness [5, §4.5].

(f) The *twisted basic* superquasigroup $T_4 = \{\pm 1\} \uplus \{\pm i\}$ is the signed superquasigroup with multiplication table

	1	i
1	1	$-i$
i	$-i$	-1

on the transversal $\{1\} \uplus \{i\}$.

(g) The *twisted Boolean* superquasigroup $W_4 = \{\pm 1\} \uplus \{\pm i\}$ is the signed superquasigroup with multiplication table

	1	i
1	1	$-i$
i	$-i$	1

on the transversal $\{1\} \uplus \{i\}$.

(h) The *right Boolean* superquasigroup $B_4^{/} = \{\pm 1\} \uplus \{\pm i\}$ is the signed supergroup with multiplication table

	1	i
1	1	$-i$
i	i	-1

on the transversal $\{1\} \uplus \{i\}$.

(i) The *left Boolean* superquasigroup $B_4^{\backslash} = \{\pm 1\} \uplus \{\pm i\}$ is the signed supergroup with multiplication table

	1	i
1	1	i
i	$-i$	-1

on the transversal $\{1\} \uplus \{i\}$.

The right and left Boolean superquasigroups are also chiral.

Right and left versions of the chiral superquasigroups mentioned above have multiplication tables which are reflections of one another along the diagonals of the table. The other quasigroups listed are symmetric along the diagonal. That is, right and left versions represent the opposite quasigroup with multiplication $a^{op}b^{op} = (ba)^{op}$.

Theorem 3.10. *The only signed superquasigroups up to superquasigroup isomorphism of order* 4 *with non-trivial grading and* 1 *idempotent are listed above. These superquasigroups are also non-supeisomorphic to one another.*

Proof. As there are only 2 loops of order 4, both of which are groups [12], each basic signed quasigroup is isotopic to either $\mathbb{Z}/_4$ or $(\mathbb{Z}/_2)^2$. If a signed superquasigroup has non-trivial grading, the element $\pm i$ must have odd polarity, and thus must square to an even element. The only even elements available are ± 1. In addition, $1i$ and $i1$ must equal $\pm i$, since i is of odd and 1 of even polarity. As 1^2 is required to be 1, all 8 possibilities are listed above.

The above are pairwise non-superisomorphic to one another. It is clear that B_4 and C_4 are not superisomorphic to any other basic quasigroup, as they are groups and the other basics are not. $C_4^{/}$, $C_4^{\backslash}$ and T_4 are isotopic to C_4 while $B_4^{/}$, $B_4^{\backslash}$ and W_4 are isotopic to B_4 and thus the first three are not isomorphic to the second three, hence not superisomorphic.

Finally, to preserve parity, i in each of these superquasigroups must be sent to $\pm i$, and since 1 is the only idempotent element, 1 must be mapped to itself and thus -1 must be mapped to -1, as that is the only place it can be sent. Thus, any superisomorphism will satisfy $(i1)^{\phi} = i^{\phi}1$. If $\phi : C_4^{/} \to C_4^{\backslash}$, then $i^{\phi} = (i1)^{\phi} = i^{\phi}1 = -i^{\phi}$, which is impossible. The same reasoning holds for an isomorphism between $B_4^{/}$ and $B_4^{\backslash}$, since the result does not depend on the square of i.

Now W_4 is not superisomorphic to $B_4^{/}$ since suppose that it is isomorphic by $\phi : W_4 \to B_4^{/}$. Then 1 is taken to 1 and -1 to -1. Since $(-i)^\phi = (i1)^\phi = i^\phi 1 = i^\phi$, this is not an isomorphism. Chirality gives the same result for W_4 and $B_4^{\backslash}$. The same reasoning holds for T_4, $C_4^{\backslash}$ and $C_4^{/}$ since the result did not depend on the square of i. □

Remark 3.11. Note that if $1 * 1 \neq 1$ in a signed basic superquasigroup Q, $1 * 1 = -1$ and thus $-1 * 1 = 1$, $1 * (-1) = 1$ and therefore $(-1)^2 = -1$, which means that -1 fulfills the role of idempotent which 1 fulfills in a basic signed superquasigroup, and relabeling 1 to -1, -1 to 1, i to $-i$ and $-i$ to i gives an isomorphism. Thus, the above list exhausts all basic signed superquasigroups up to superisomorphism.

3.2. Signed (super)products

Definition 3.12. [5, Def. 4.13] Let $(Q, *)$ and $(P, \circ)$ be signed superquasigroups on disjoint sets. Define the *signed superproduct* $\widehat{\times}$ on the product set $Q \times P$ as a signed superquasigroup with transversal given by the product of the transversal T of Q and the transversal U of P subject to the relation $(q \otimes -p) = (-q \otimes p) = -(q \otimes p)$ and $(-q \otimes -p) = (q \otimes p)$. Let T_0 be the even part of T and T_1 the odd part. Define U_0 and U_1 similarly. Then, define multiplication on this transversal as

$$(t \otimes t') \cdot (u \otimes u') = (-1)^{|t'| \cdot |u|}(tu \otimes t'u') \tag{3.2}$$

The definitions of left and right division follow immediately:

$$(t \otimes t')/(u \otimes u') = (-1)^{(|t'|+|u'|) \cdot |u|}(t/u \otimes t'/u') \tag{3.3}$$
$$(t \otimes t')\backslash(u \otimes u') = (-1)^{|t'| \cdot (|t|+|u|)}(t\backslash u \otimes t'\backslash u') \tag{3.4}$$

Lemma 3.13. [5, Thm. 4.15] *Definition* 3.12 *defines a signed superquasigroup.*

Proposition 3.14. [5, Cor. 4.16] *The signed product of two signed superloops* (*groups*) *is a superloop* (*group*).

Proposition 3.15. *Let A and B be signed quasigroups. Then $(A \widehat{\times} B)^{\mathrm{op}}$ is isomorphic to $B^{\mathrm{op}} \widehat{\times} A^{\mathrm{op}}$.*

Proof. Let ϕ take $(a \otimes b)^{\mathrm{op}}$ to $b^{\mathrm{op}} \otimes a^{\mathrm{op}}$. This is clearly a bijection. Then $((a \otimes b)^{\mathrm{op}} \cdot (a' \otimes b')^{\mathrm{op}})^\phi = (-1)^{|b'|\,|a|}(b'b)^{\mathrm{op}} \otimes (a'a)^{\mathrm{op}}$ while $((a \otimes b)^{\mathrm{op}})^\phi((a' \otimes b'))^{\mathrm{op}}))^\phi = (b^{\mathrm{op}} \otimes a^{\mathrm{op}})(b'^{\mathrm{op}} \otimes a'^{\mathrm{op}}) = (-1)^{|b'|\,|a|}(b'b)^{\mathrm{op}} \otimes (a'a)^{\mathrm{op}}$, as required. □

The unit basic signed superquasigroup derives its name from the following property.

Proposition 3.16. [5, Prop. 4.18] *There are isomorphisms $C_2 \widehat{\times} Q \cong Q \cong Q \widehat{\times} C_2$ for any signed superquasigroup Q.*

3.3. The category of signed superquasigroups

Definition 3.17. Signed superquasigroups form a category. Let $(P,\sim)$ and $(Q,\sim)$ be signed superquasigroups with equivalence relations defined so that $x \sim y$ if and only if $x = \pm y$. Then a morphism in the category of signed superquasigroups is given by the commuting diagram of superquasigroup homomorphisms

$$\begin{array}{ccc} Q & \twoheadrightarrow & Q/\sim \\ \downarrow & & \downarrow \\ P & \twoheadrightarrow & P/\sim \end{array}$$

with the added condition that $\phi(-x) = -\phi(x)$.

The unit basic signed superquasigroup acts as an identity element for $\widehat{\times}$. The empty signed superquasigroup acts as an initial object.

If there exist maps $\phi : A \to B$ and $\psi : A \to C$, then the map $\phi\widehat{+}\psi : A \to B\widehat{\times}C$ can be defined such that if $a \in A$, then its image under $\phi\widehat{+}\psi$ is $a^\phi \otimes a^\psi$. This is indeed a signed superquasigroup morphism, since

$$(a^\phi \otimes a^\psi)(a'^\phi \otimes a'^\psi) = (-1)^{|a'^\phi|\,|a^\psi|} a^\phi a'^\phi \otimes a^\psi a'^\psi = (-1)^{|a'|\,|a|}(aa')^\phi \otimes (aa')^\psi.$$

From this it might be concluded that $\widehat{\times}$ is a categorical product. Unfortunately, $\widehat{\times}$ fails to be a product, because the required projections do not exist. For $\pi_1 : X\widehat{\times}Y \to X$ to be a surjective map, it is necessary that there exist a congruence such that $a \otimes b$ is identified with $a' \otimes b$ whenever $a = a'$. But if this is the case then $a \otimes (-b)$ must be identified with $a \otimes b$, and thus $-a \otimes b$ is identified with $a \otimes b$. But then the image of X is the quotient $X/\sim$ from Definition 3.17, which is not necessarily a subquasigroup of X, and thus the image is not contained in X and this is not a projection. Thus $\widehat{\times}$ fails to be a product in the category of signed superquasigroups, and for the same reason will fail to be one in the category of signed superpiques or signed superloops. It is also the case that $\times$ is not a product, since projection maps will fail to satisfy $(-x)^\phi = -x^\phi$. However, the product $A \times B$ is a signed superquasigroup, because it is possible to choose either the negation of A or the negation of B to be the negation of the whole superquasigroup. Without loss of generality, choose the negation in A to be the negation of the whole signed superquasigroup. Then the projection functor $\pi_B : A \times B \to B$ does not preserve negation, since $((-a) \times b)^{\pi_B} = b$, while $-(a \times b)^{\pi_B} = -b$.

The more useful construction takes $\phi : A \to B$ and $\psi : C \to D$ and gives the map $\phi\widehat{\times}\psi : A\widehat{\times}C \to B\widehat{\times}D$, taking $a\widehat{\times}c \in A\widehat{\times}C$ to $a^\phi\widehat{\times}c^\psi$. This is indeed a signed superquasigroup homomorphism, since

$$\begin{aligned} &(a \otimes c)^{\phi\widehat{\times}\psi}(a' \otimes c')^{\phi\widehat{\times}\psi} = (a^\phi \otimes c^\psi)(a'^\phi \otimes c'^\psi) \\ =&(-1)^{|a'^\phi|\,|c^\psi|} a^\phi a'^\phi \otimes c^\psi c'^\psi \\ =&(-1)^{|a'|\,|c|}(aa')^\phi \otimes (cc')^\psi = ((a \otimes c)(a' \otimes c')^{\phi\widehat{\times}\psi}. \end{aligned}$$

Lemma 3.18. *There is a natural isomorphism in the category of signed superquasigroups between* $(A\widehat{\times}B)\widehat{\times}C$ *and* $A\widehat{\times}(B\widehat{\times}C)$.

Proof. If A, B and C are signed superquasigroups, with $a, a' \in A$, $b, b' \in B$ and $c, c' \in C$ then $(A\widehat{\times}B)\widehat{\times}C$ has multiplication

$$((a \otimes b) \otimes c)((a' \otimes b') \otimes c') = (-1)^{|c|\,|a'|+|c|\,|b'|+|b|\,|a'|}(aa' \otimes bb') \otimes cc'$$

while $A\widehat{\times}(B\widehat{\times}C)$ has multiplication

$$(a \otimes (b \otimes c))(a' \otimes (b' \otimes c')) = (-1)^{|a'|\,|c|+|a'|\,|b|+|c|\,|b'|}aa' \otimes (bb' \otimes cc'),$$

and the map $\Xi_{A,B,C}$ taking $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$ is an isomorphism. This isomorphism is natural, since the following diagram commutes:

$$\begin{array}{ccc}
(A\widehat{\times}B)\widehat{\times}C & \xrightarrow{\Xi_{A,B,C}} & A\widehat{\times}(B\widehat{\times}C) \\
\Big\downarrow{\scriptstyle f_A\widehat{\times}(f_B\widehat{\times}f_C)} & & \Big\downarrow{\scriptstyle f_A\widehat{\times}(f_B\widehat{\times}f_B)} \\
(D\widehat{\times}E)\widehat{\times}F & \xrightarrow{\Xi_{D,E,F}} & D\widehat{\times}(E\widehat{\times}F)
\end{array}$$

□

The unit basic supergroup I acts as the identity of the signed product and it is straightforward to show that $(\widehat{\times}, I)$ makes the category of signed superquasigroups into a monoidal category [8, §VII.1].

Proposition 3.19. *The signed product is a tensor product on the category of signed superquasigroups. That is, the signed superproduct turns this category into a monoidal category.*

Proof. Let A, B, C and D be signed superquasigroups, with $a, a' \in A$, $b, b' \in B$, $c, c' \in C$ and $d, d' \in D$. For the signed superproduct to yield a monoidal category, it is necessary to show algebraically that any association of A, B, C and D will give the same sign when multiplication is carried out in their products.

Firstly, $(a \otimes (b \otimes (c \otimes d)))(a' \otimes (b' \otimes (c' \otimes d')))$ is equal to

$$(-1)^{(|b|+|c|+|d|)|a'|+(|c|+|d|)|b'|+|d||c'|}(aa' \otimes (bb' \otimes (cc' \otimes dd'))).$$

Dually, $(((a \otimes b) \otimes c) \otimes d)(((a' \otimes b') \otimes c') \otimes d')$ is equal to

$$(-1)^{|d|(|a'|+|b'|+|c'|)+|c|(|a'|+|b'|)+|b||a'|}(((aa' \otimes bb') \otimes cc') \otimes dd').$$

In addition, $(a \otimes ((b \otimes c) \otimes d))(a' \otimes ((b' \otimes c') \otimes d'))$ is equal to

$$(-1)^{(|b|+|c|+|d|)|a'|+|d|(|b'|+|c'|)+|c||b'|}(aa' \otimes ((bb' \otimes cc') \otimes dd').$$

Dually, $((a \otimes (b \otimes c)) \otimes d)((a' \otimes (b' \otimes c')) \otimes d')$

$$= (-1)^{|d|(|a'|+|b'|+|c'|)+(|b|+|c|)|a'|+cb'}((aa' \otimes (bb' \otimes cc')) \otimes dd').$$

Finally, $((a \otimes b) \otimes (c \otimes d))((a' \otimes b') \otimes (c' \otimes d'))$ gives us a product equal to

$$(-1)^{(|c|+|d|)(|a'|+|b'|)+|d||c'|+|b||a'|}((aa' \otimes bb') \otimes (cc' \otimes dd')).$$

Thus, the power of -1 is equal in all cases, as can be seen by distributing the exponents of -1. Therefore, if 1_X is the X component of the identity natural isomorphism between the identity functor and itself and $\Xi_{X,Y,Z}$ is the natural isomorphism from Lemma 3.18 then the following diagram commutes:

$$
\begin{array}{ccccc}
A\widehat{\times}(B\widehat{\times}(C\widehat{\times}D)) & \xrightarrow{\Xi_{A,B.C\widehat{\times}D}} & (A\widehat{\times}B)\widehat{\times}(C\widehat{\times}D) & \xrightarrow{\Xi_{A\widehat{\times}B,C,D}} & ((A\widehat{\times}B)\widehat{\times}C)\widehat{\times}D \\
\Big\downarrow{\scriptstyle 1_A\widehat{\times}\Xi_{B,C,D}} & & & & \Big\downarrow{\scriptstyle \Xi_{A,B,C}\widehat{\times}1_D} \\
A\widehat{\times}((B\widehat{\times}C)\widehat{\times}D) & & \xrightarrow{\Xi_{A,B\widehat{\times}C,D}} & & (A\widehat{\times}(B\widehat{\times}C))\widehat{\times}D
\end{array}.
$$

Now consider the isomorphism maps from $\lambda_A : A\widehat{\times}I \to A$, $\rho : I\widehat{\times}B \to B$ which exist by Proposition 3.16. Then the following diagram commutes:

$$
\begin{array}{ccc}
A\widehat{\times}(I\widehat{\times}B) & \xrightarrow{\Xi_{A,I,B}} & (A\widehat{\times}I)\widehat{\times}B \\
& {\scriptstyle 1_A\widehat{\times}\rho_B}\searrow \quad \swarrow{\scriptstyle \lambda_A\widehat{\times}1_B} & \\
& A\widehat{\times}B &
\end{array}
$$

because the negative sign can be moved freely from the first term in the signed superproduct to the second. ☐

Finally, if instead of taking general signed superquasigroups, the subcategory of signed superpiques is considered, then morphisms in that subcategory can be defined by the commutative diagram

$$
\begin{array}{ccccccccc}
\{1\} & \hookrightarrow & \{\pm 1\} & \hookrightarrow & Q & \twoheadrightarrow & Q/\{\pm 1\} & \twoheadrightarrow & \{1\} \\
\Big\Downarrow & & \Big\Downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\Downarrow \\
\{1\} & \hookrightarrow & \{\pm 1\} & \hookrightarrow & P & \twoheadrightarrow & P/\{\pm 1\} & \twoheadrightarrow & \{1\}.
\end{array}
$$

The same categorical properties proved in the category of signed superquasigroups are inherited by the category of signed superpiques. In addition, the unit basic supergroup becomes an initial object. Similar reasoning derives monoidal categories of signed superloops and signed supergroups.

4. Signed superproducts

4.1. Signed superproducts of basic signed supergroups

The product $B_4\widehat{\times}B_4$ has multiplication table with transversal

	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$1\otimes 1$	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$i\otimes i$	$i\otimes i$	$-1\otimes 1$	$i\otimes 1$	$-1\otimes i$
$1\otimes i$	$1\otimes i$	$-i\otimes 1$	$1\otimes 1$	$-i\otimes i$
$i\otimes 1$	$i\otimes 1$	$1\otimes i$	$i\otimes i$	$1\otimes 1$

and is therefore isomorphic to the dihedral group on 4 elements, D_4 [5, §4.6.2].

On the other hand, the transversal of the multiplication table of $C_4\widehat{\times}C_4$ is

	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$1\otimes 1$	$1\otimes 1$	$i\otimes i$	$1\otimes i$	$i\otimes 1$
$i\otimes i$	$i\otimes i$	$-1\otimes 1$	$-i\otimes 1$	$1\otimes i$
$1\otimes i$	$1\otimes i$	$i\otimes 1$	$-1\otimes 1$	$-i\otimes i$
$i\otimes 1$	$i\otimes 1$	$-1\otimes i$	$i\otimes i$	$-1\otimes 1$

,

and is therefore isomorphic to the quaternion group Q_8 [5, §4.6.1].

The product of B_4 and C_4 is also D_4 [5, §4.7.1].

It turns out that up to isotopy, these are the only two products of basic signed superquasigroups of order 4, as will be shown in Section .

4.1. Relation to Clifford algebras

Recall the definition of a Clifford algebra over $\mathbb{R}$.

Definition 4.1. The Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$ is generated by p generators squaring to -1 and q generators squaring to 1. Distinct generators mutually anticommute.

Repeatedly taking the supertensor product of C_4 or B_4 gives groups closely related to Clifford algebras over the reals.

Theorem 4.2. $\mathrm{Cl}_{n,m}(\mathbb{R}) = \big(\widehat{\times}_{i=1}^{n} C_4\big)\,\widehat{\times}\,\big(\widehat{\times}_{i=1}^{m} B_4\big)\mathbb{R}/I$ *where I is the ideal generated by identifying -1 in the product with $-1 \in \mathbb{R}$.*

Proof. The elements of the signed product $\bigotimes_{j=1}^{n} C_4 \,\widehat{\times} \bigotimes_{j=1}^{m} B_4$ can be represented as ordered $n+m$-tuples by Lemma 3.18. The whole signed super group is generated by the set of tuples with 1 nonidentity entry equal to i and $n+m-1$ entries equal to 1, Since groups are isomorphic to their opposites, Proposition 3.15 and Lemma 3.18 imply that the signed product of supergroups can be rearranged in any order. Thus, it is sufficient to examine the product of i_l and i_{l+1}, where i_l is the element with the l^{th} term equal to i and all others equal to 0. Then $i_l i_{l+1} = -i_{l+1} i_l$ and i_l, i_{l+1} square to ± 1 by Section 4.1. Thus, a generating set satisfies the conditions imposed on the generating set of a real Clifford algebra. It remains to show that we have the correct Clifford algebra. Since real Clifford algebras are totally determined by the number of generators which square to ± 1, and the generators with i from C_4's square to -1 and those from B_4's square to 1, $\big(\widehat{\times}_{i=1}^{n} C_4\big)\,\widehat{\times}\,\big(\widehat{\times}_{i=1}^{m} B_4\big)\mathbb{R}/I \cong \mathrm{Cl}_{n,m}(\mathbb{R})$. □

5. Standard signed superquasigroups

Definition 5.1. Call a finite signed superquasigroup Q standard if there is a short exact sequence $\{1\} \to \{\pm 1\} \to Q \to L \to \{1\}$, where L is a superloop and each map is superquasigroup homomorphism.

The group $\{\pm 1\}$ provides the sign to Q while ensuring the existence of a pointed idempotent as the image of 1, while L is the standard signed supergroup interpreted without signs

Note that standard superquasigroups can be loops but do not have to be, since $1 \cdot x$ may equal -1. They are loops up to the addition of a negative sign.

Lemma 5.2. *All signed basic superquasigroups are standard.*

Proof. Each of them is an extension of an order 2 supergroup with 1 even and i odd. That is, they have short exact sequence $\{1\} \to \{\pm 1\} \to Q \to \mathbb{Z}/_2 \to \{1\}$. ☐

Lemma 5.3.

1. *The signed superproduct of two standard signed superquasigroups is again standard.*
2. *Both standard superquasigroups appear as signed subquasigroups within their signed superproduct.*

Proof. 1. If $\phi : Q \to L$, $\psi : P \to M$ are surjective superquasigroup morphisms satisfying the exact sequences $\{1\} \to \{\pm 1\} \to Q \to L \to \{1\}$ and $\{1\} \to \{\pm 1\} \to P \to M \to \{1\}$ then define the map $f : Q \widehat{\times} P \to L \times M$ to take $\pm x \otimes y \in Q \widehat{\times} P$ to $x \times y$. This gives the required surjection and the map which takes $1 \in \{\pm 1\}$ to $e \otimes e \in Q \widehat{\times} P$ and -1 to $-e \otimes e \in Q \widehat{\times} P$ gives the required injection.

2. We show $Q \otimes e$ is a standard sub-superquasigroup isomorphic to Q. The proof that $e \otimes P$ is a standard sub-superquasigroup isomorphic to P is identical. Note that for $q \in Q$ both $(q \otimes e)$ and $(-q \otimes e)$ are in $Q \otimes e$, as is $(e \otimes e)$, so if any element is contained in $Q \otimes e$, so is its negation, as well as the pointed idempotent $e \otimes e$. Now,

$$(q \otimes e)(q' \otimes e) = (-1)^0 qq' \otimes e \in (Q, e) \tag{5.5}$$

$$(q \otimes e)(e \otimes e) = (qe \otimes e) \in (Q, e) \tag{5.6}$$

$$(e \otimes e)(q \otimes e) = (eq \otimes e) \in (Q, e) \tag{5.7}$$

$$(q \otimes e)/(q' \otimes e) = (-1)^{(|e|+|e|)\cdot|q'|}(q/q' \otimes e) = q/q' \otimes e \in (Q, e) \tag{5.8}$$

$$(q \otimes e)\backslash(q' \otimes e) = (-1)^{|e|\cdot(|q|+|q'|)}(q\backslash q' \otimes e) = q\backslash q' \otimes e \in (Q, e) \tag{5.9}$$

and the set (Q, e) is a subquasigroup. ☐

Corollary 5.4. *The collection of standard superquasigroups is a monoidal category with* $\widehat{\times}$ *as its tensor product and the unit basic supergroup as its identity element.*

Definition 5.5. Define the *signed isotopy* on $(Q, \circ)$ by $(q \circ e) \circ (e \circ p) = q \cdot p$, where e is the pointed idempotent of Q. The resulting loop with multiplication $\cdot$ is labeled Q_S.

Note what this does. If $q \circ e = -q$, then $q \cdot e = (q \circ e) \circ e = q$ while if $e \circ q = -q$, $e \cdot q = e \circ (e \circ q) = q$. On the other hand, if $q \circ e = q$, the signed isotopy does nothing on the left, while it does nothing on the right if $e \cdot q = q$. Thus Q_S is a loop. The signed isotopy is clearly a superisotopy, since a and $-a$ have the same parity. This isotopy is unique by definition. Since $ee = e$, e is sent to itself by all three maps in the isotopy. Similarly $e(-e) = (-e)e = -e^2 = -e$. A standard superquasigroup is *signed isotopic* to a loop if it is isotopic to that loop via the signed isotopy.

Definition 5.6. Let $(Q_S, \Box)$ be the loop signed isotopic to $(Q, \cdot)$ and $(P_S, *)$ be the loop signed isotopic to $(P, \circ)$. Then define an isotopy from $(Q_S \widehat{\times} P_S, \cdot \widehat{\times} *)$ to $(Q \widehat{\times} P, \Box \widehat{\times} \circ)$ by $(q \otimes p)(\Box \widehat{\times} *)(q' \widehat{\times} p') = ((q \cdot e) \cdot (e \cdot p), (p \circ e) \circ (e \circ p'))$. Call this the signed isotopy product of Q and P.

Theorem 5.7. *In the above definition $Q_S \widehat{\times} P_S$ is a loop. Furthermore, it is the same loop as $(Q \widehat{\times} P)_S$.*

Proof. To show this, note first that $(e,e)(\Box\widehat{\times}*)(q,p) = ((q \cdot e) \cdot (e \cdot q'), (p \circ e) \circ (e \circ p') = (q,p) = (q,p)(\Box\widehat{\times}*)(e,e)$. It is clearly the signed isotopy from $(Q\widehat{\times}P)_S$ to $Q\widehat{\times}P$ since both the left and right bijection only flip signs, and how they flip signed is determined solely by how the idempotents multiply with elements of $Q\widehat{\times}P$. □

Corollary 5.8. *Arbitrary signed superproducts of basic standard signed superquasigroups are isotopic to supergroups.*

Proof. Since every basic signed superquasigroup is a standard superquasigroup isotopic to groups C_2, C_4 or B_4, and products of standard superquasigroups are standard superquasigroups, the products of these basic signed superquasigroups will be isotopic to signed superproducts of C_2, C_4 or B_4, hence will be isotopic to groups. □

Note that not every signed superproduct of signed superquasigroups will be isotopic to a group, because not every standard superquasigroup is isotopic to a group. A signed product of two standard superquasigroups not isotopic to groups need not be isotopic to a group. The result above only applies to the basic signed superquasigroups listed in Section , as they are isotopic to groups.

Corollary 5.9. *signed superproducts of order 4 basic signed superquasigroups are isotopic to Q_8 or D_4.*

Proof. The result follows immediately from Section 4.1 and Lemma 3.10. Since all basic signed superquasigroups are signed isotopic to basic signed supergroups, and the signed product of the two order 4 basic signed superquasigroups is either Q_8 or D_4, the products of order 4 basic signed superquasigroups are isotopic to Q_8 or D_4. □

6. Hedral loops

We define a *hedral loop* to be a standard loop which is an extension of the Klein-4 group. The Klein-4 group is viewed here as non-trivially graded modulo-2. The resulting signed superquasigroups will be of order 8.

Definition 6.1. A *hedral loop* L (after the dihedral group) is a standard signed superloop which satisfies the exact sequence $\{1\} \to \{\pm 1\} \to L \to (\mathbb{Z}/_2 \times \mathbb{Z}/_2) \to \{1\}$.

The dihedral group D_4, the quaternion group Q_8 and Smith and Im's quatedral loop [5, §4.7.2] are all examples. To assign negative signs to a multiplication table, we define the following function.

Definition 6.2. Let $F : (\mathbb{Z}/_2 \times \mathbb{Z}/_2)^2 \to \mathbb{Z}/_2$ take pairs of elements in the Klein-4 group to 0 if the product is to be positive and to 1 if it is to be negative, assigning a negative or positive value to each position in the multiplication table of $(\mathbb{Z}/_2 \times \mathbb{Z}/_2)$. Define the multiplication for $(-1)^a x$, $(-1)^b y$, $x, y \in (\mathbb{Z}/_2)^2$, $a, b \in \{0, 1\}$ by $(-1)^{xyF+a+b} x \circ y$, where $\circ$ is the multiplication of the Klein-4 group.

It is clear that this new multiplication table is that of a hedral loop if and only if $(1,x)^F = (x,1)^F = x$ for all $x \in (\mathbb{Z}/_2 \times \mathbb{Z}/_2)$.

Proposition 6.3. *Hedral loops have the following two properties:*

1. *They are power-associative.*
2. *They have two sided inverses.*

Proof. Since any element generates an abelian subgroup, the result is obvious. For the second result, note that if $x^2 = 1$, x is its own inverse, and clearly commutes with itself. If $x^2 = -1$, then $-x$ is its inverse, and $-x$ commutes with x for all x. □

Lemma 6.4. *Let $(L,*)$ and $(M,\circ)$ be loops and suppose that L has two sided inverses. For any principal isotopy $(\alpha,\beta) : M \to L$, the principal isotopy takes the form $(gR(1^\beta)^{-1})(hL(1^\alpha)^{-1})$.*

Proof. Note that $1^\alpha * g^\beta = 1 \circ g = g = g \circ 1 = g^\alpha * 1^\beta$ and thus $g^\alpha = gR(1^\beta)^{-1}$ while $g^\beta = L(1^\alpha)^{-1}g$. Thus, $g^\alpha h^\beta = (gR(1^\beta)^{-1})(hL(1^\alpha)^{-1})$. □

Proposition 6.5. *Let $(Q,*)$ and $(Q,\cdot)$ be hedral loops with positive non-identity elements x, y and z. Suppose that $*$ and $\cdot$ agree when multiplying $\pm z$ with any other element but satisfy $x * y = -x \cdot y$, $y * x = -y \cdot x$. Then the two loops are isotopic.*

Proof. Let g take ± 1, $\pm x$ and $\pm y$ to themselves, z to $-z$ and $-z$ to z. Let h send ± 1 to itself and send all other elements to their negatives. Then $x^g * z^g = -x * z = \pm y$; while $(x \cdot z)^h = (x * z)^h = -x * z$ because $x \cdot z = x * z = \pm y$. By the same reasoning $(z \cdot x)^h = z^g * x^g$, and this reasoning also holds with y in place of x. In addition, $x^g * y^g = x * y$ while $(x \cdot y)^h = (-x * y)^h = x * y$. The result with y and x swapped is proven identically. Finally $x^g * x^g = (x \cdot x)^h$ and similarly for y and z. Thus, $x^g * y^g = (x \cdot y)^h$ □

Corollary 6.6. *A hedral loop is isotopic to its opposite.*

6.1. Examples

Let i, j and ij be the three nonidentity elements of the Klein-4 group G. Define G_F to be the loop resulting from the function F of Definition 6.2. This loop will be a hedral loop of order 8. Its multiplication will be written $\circ_F$

Example 6.7. Given F such that $(1,a)^F = (a,1)^F = 0$ for all $a \in G$ (recall that G is the Klein-4 Group), $(i,j)^F = (j,ij)^F = (i,ij)^F = 0$, $(j,i)^F = (j,ij)^F = (ij,i)^F = 1$ and $(i,i)^F = (j,j)^F = (ij,ij)^F = 1$, this F makes G_F isomorphic to the quaternion group, where $\circ_F$ is defined by $x \circ_F y = (-1)^{(x,y)^F} x \circ y$.

Example 6.8. Given F such that $(1,a)^F = (a,1)^F = 0$ for all $a \in G$, $(i,j)^F = (j,ij)^F = (i,ij)^F = 0$, $(j,i)^F = (j,ij)^F = (ij,i)^F = 1$, $(i,i)^F = (ij,ij)^F = 0$ and $(j,j)^F = 1$, then $(G_F,\circ_F)$ is the dihedral group.

It is now possible to define several more loops as examples of this procedure.

Example 6.9. Suppose a, b and c are the non-identity elements of the Klein-4 group. Let $(1,x)^F = (x,1)^F = 0$ for all $x \in G$ and $(b,a)^F = (a,c)^F = (c,b)^F = (b,c)^F = 0$. Let all other elements $G \times G$ be mapped to 1. Then G_F is a loop such that a commutes with c and b commutes with c, but b and a anticommute. This is indeed a loop because $1x = x1$ for all $x \in G_F$. Some of its non-identity elements commute and some anticommute, so this loop of order 8 is not isotopic or isomorphic to either D_4, Q_8, or any abelian group and thus is not a group. It has multiplication table.

$*$	1	-1	a	$-a$	b	$-b$	c	$-c$
1	1	-1	a	$-a$	b	$-b$	c	$-c$
-1	-1	1	$-a$	a	$-b$	b	$-c$	c
a	a	$-a$	-1	1	$-c$	c	b	$-b$
$-a$	$-a$	a	1	-1	c	$-c$	$-b$	b
b	b	$-b$	c	$-c$	-1	1	a	$-a$
$-b$	$-b$	b	$-c$	c	1	-1	$-a$	a
c	c	$-c$	b	$-b$	a	$-a$	-1	1
$-c$	$-c$	c	$-b$	b	$-a$	a	1	-1

Example 6.10 (Quatedral loop)**.** Instead of using a, b and c, we use i, e and ie for the following example, following [5, 4.7.2] Let $(1,x)F = (x,1)F = 0$ for all $x \in G$ and $(i,e)F = (e,i)F = (i,ie)F = (e,e)F = 0$, with all other (x,y) such that $(x,y)F = 1$. Then G_F is a loop such that e commutes with ie and i commutes with e but i anticommutes with ie. This is indeed a loop because $1x = x1$ for all $x \in G_F$. Some of its non-identity elements commute and some anticommute, so this loop of order 8 is not isotopic or isomorphic to the groups D_4, Q_8, or any Abelian group and thus is not a group. It is, however, isomorphic to the Quatedral loop in [5, 4.7.2]. Its multiplication table, as given in [5, 4.7.2], is

$*$	1	i	-1	$-i$	e	ie	$-e$	$-ie$
1	1	i	-1	$-i$	e	ie	$-e$	$-ie$
i	i	-1	$-i$	1	ie	e	$-ie$	$-e$
-1	-1	$-i$	1	i	$-e$	$-ie$	e	ie
$-i$	$-i$	1	i	-1	$-ie$	$-e$	ie	e
e	e	ie	$-e$	$-ie$	1	$-i$	-1	i
ie	ie	$-e$	$-ie$	e	$-i$	-1	i	1
$-e$	$-e$	$-ie$	e	ie	-1	i	1	$-i$
$-ie$	$-ie$	e	ie	$-e$	i	1	$-i$	-1

6.1. Quasigroup character theory

Character tables for finite quasigroups are defined using their multiplication groups as outlined in [9], [10, §6], [11]. The definition given in [10, §6] uses, and is completely determined by, the orbits of the multiplication group action on $Q \times Q$, or equivalently, an orbit of (e, Q) for some $e \in Q$ under the action of the stabilizer of e in G. To construct these tables, it is necessary to find the *centralizer ring* $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}Q)$, where G is the multiplication group of the quasigroup and $\mathbb{C}Q$ is the quasigroup algebra of Q over $\mathbb{C}$

viewed as a right module over $\mathbb{C}G$. The centralizer ring, usually written $V(G, Q)$, is, as stated earlier, the algebra $\mathrm{End}_{\mathbb{C}G}(\mathbb{C}Q)$. That is, it is the set of $\mathbb{C}G$ endomorphisms of $\mathbb{C}Q$.

This algebra can be shown to be commutative [10, §6.3] and to be generated by a collection of matrices, each associated with an orbit of the diagonal G-action on $Q \times Q$. These orbits are called *quasigroup conjugacy classes*. The matrix A_S is the adjacency matrix of the quasigroup conjugacy class S, which has entry $a_{ij} = 1$ if $(q_i, q_j) \in S$ and 0 otherwise [10, §6.1,5]. Since quasigroup conjugacy classes are orbits of a group action, and orbits are disjoint, these matrices will be linearly independent. The A_S's are one set of standard generators of the centralizer ring. The value n_i associated with the i^{th} conjugacy class C_i is the cardinality of the set $\{q : (x, q) \in C_i\}$.

Since the matrices A_i commute, each fixes the eigenspaces of the others, and it is possible to partition $\mathbb{C}Q$ into orthogonal subspaces which are eigenspaces of every one of the conjugacy classes under the multiplication group action. Each of these subspaces is 1 dimensional and has as a standard basis element an element E_i, where $E_1 = \frac{1}{n}\sum_{i=1}^{s} A_i$, where s is the number of orbits and n the order of Q. This matrix has all entries equal to 1. The E_i are chosen so that $E_iE_j = \delta_{ij}E_i$. That is, the E_i are idempotent and mutually orthogonal. The trace of each E_i is given by f_i. Let $E_i = \sum_{j=1}^{s} \eta_{ij}A_j$ and $A_i = \sum_{j=1}^{s} \xi_{ij}E_j$. Then the character table $\Psi = (\psi_{ij})$ will have entries $\psi_{ij} = \frac{\sqrt{f_i}}{n_j}\xi_{ji} = \frac{n}{\sqrt{f_i}}\overline{\eta_{ij}}$. The value n here is the cardinality of the quasigroup. The standard orthogonality relations hold for these quasigroup character tables (see [10, §6.7]. The equality of the two expressions for ψ_{ij} is shown in Lemma 6.2 of the preceding section of the same source.). The class functions $\psi_i(x, y) = \psi_{ij}$ form what are called basic characters of Q, since they are fixed on the orbits of $Q \times Q$ under the action of the multiplication group. They form an orthogonal basis for such class functions under the inner product $\frac{1}{n^2}\sum_{(x,y)\in Q^2} \theta(x, y)\phi(y, x)$ [11].

Let's take a simple example with the group $\mathbb{Z}/_3$ represented additively (this is exercise 6 of Section 2.1 in [11]). Since it is abelian, it is isomorphic to its multiplication group. The adjacency matrix of A_1 of $(0, 0)$ is the diagonal matrix. This is always the case for the diagonal entries in $Q \times Q$, which are necessarily in the same conjugacy class, which always satisfies $n_i = 1$. In abelian groups, every conjugacy class in $Q \times Q$ will necessarily satisfy $n_i = 1$, as any x in a group $(1, x)$ is in a different conjugacy class from $(1, x')$ whenever $x \neq x'$. We can find $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, while $A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. This is just because $(0, 1)$ acted on by $\mathbb{Z}/_3$ gives $(1, 2)$ and $(2, 0)$ as conjugates by adding 1 and 2. To find A_3, begin with $(0, 2)$ instead. In fact, this same sort of reasoning means that any abelian group will have adjacency matrices for its conjugacy classes which are permutation matrices, since if $(0, a)$ is in a conjugacy class, $(0, b)$ is in that conjugacy class if and only if $a = b$. The matrix E_1 will always be the matrix with all entries equal to $\frac{1}{n}$. This will always occur among the E_i's so it is standard to make it the first of the E_i's. The second, E_2, is equal to $\frac{1}{3}\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}$, while $E_3 = \frac{1}{3}\begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}$, where ω is a primitive third root of unity. Each E_i is idempotent and orthogonal. It is also clear that $\frac{A_1+A_2+A_3}{3} = E_1$ and $E_1 + E_2 + E_3 = A_1$. So $\xi_{1j} = 1$ and $\eta_{1j} = \frac{1}{3}$ Since $f_i = 1$ for all E_i, and $n_i = 1$ for all A_i. $\psi_{i1} = \frac{\sqrt{f_i}}{n_1}\xi_{1i} = \frac{\sqrt{1}}{1}1$ while $\psi_{j1} = \frac{n}{\sqrt{f_1}}\overline{\eta_{1j}} = \frac{3}{\sqrt{1}}\frac{1}{3} = 1$. This shows that the first row and column of the character table have all entries equal to 1, which corresponds exactly with the character table of $\mathbb{Z}/_3$. Now $E_2 = \frac{1}{3}A_1 + \frac{\omega}{3}A_2 + \frac{\omega^2}{3}A_3$ and $E_3 = \frac{1}{3}A_1 + \frac{\omega^2}{3}A_2 + \frac{\omega}{3}A_3$. Then $\psi_{22} = \frac{3\omega^2}{3} = \omega^2$, $\psi_{23} = \frac{3\omega}{3} = \omega$. The last row is $\psi_{32} = \omega$ and $\psi_{33} = \omega^2$, as required.

This definition can be shown to be equivalent to the ordinary definition of group representations [11, §2.3.2] when the quasigroup in question is a group. However, we will give a proof for abelian groups only.

Proposition 6.11. *If H is a group, the orbits of (e, H) under the action of the the subgroup of the multiplication group G of H $G_e$which fixes the identity consist only of conjugations.*

Proof. Since H is associative, any collection of right and left multiplications in G can be re-expressed in the form $L(x)R(y)$. The only such elements fixing e can be written in the form $L(x^{-1})R(x)$. Thus, the elements fixing e are conjugations, and the orbit of (e, h) under G_e consists of conjugates of elements of the form (e, ghg^{-1}), for $g \in H$. □

Theorem 6.12. *If H is an abelian group, its character table computed in the quasigroup manner is the same as its character table computed in the ordinary group representation manner.*

Proof. Suppose first that G is the cyclic abelian group of size n. Then the multiplication group of G is G, $n_i = 1$ for every conjugacy class C_i. There will be n such adjacency matrices, and their sum will be nE_1. Since these are distinct permutation matrices with no entries in common, they will be linearly independent. The entries of A_k will satisfy $(A_k)_{ij} = \delta_{i+k-1,j}$, where addition is interpreted modulo n and δ is the Kronecker δ. That is, non-zero entries occur when $i+k+1 = j \bmod n$ or equivalently, when $j-i = k+1 \bmod n$

The idempotent mutually orthogonal matrix E_k will have entries $(e_k)_{ij} = \frac{1}{n}(\omega^{k-1})^{j-i}$, where ω is any primitive n^{th} root of unity. We confirm that these E_k's fulfill the requirements imposed on them in the definition. Note first that E_1 has all entries equal to $\frac{1}{n}$. The entry $(E_kE_k)_{ij} = \frac{1}{n^2}\sum_{m=1}^{n}(\omega^{k-1})^{m-i}(\omega^{k-1})^{j-m} = \frac{n}{n^2}(\omega^{k-1})^{j-i} = (E_k)_{ij}$, as required for idempotent matrices.
Let $k \neq k'$. Then $(E_kE_{k'})_{ij} = \frac{1}{n^2}\sum_{m=1}^{n}(\omega^{k-1})^{m-i}(\omega^{k'-1})^{j-m}$. Simplifying, this yields the sum $\frac{1}{n^2}\sum_{m=1}^{n}\omega^{k's-s-kr+r+m(k-k')}$. The sum will cycle through roots of unity. If $k-k'$ is coprime with n, it will cycle once through n^{th} roots of unity. If instead, $k-k'$ has common divisor l with n, it will cycle through $\left(\frac{n}{l}\right)^{\text{th}}$ roots of unity, and will do so l times. In either case, all roots of unity for some value will occur the same number of times, and the sum of all of them will be 0. The E_i's are linearly independent, since suppose that E_k is equal to a linear combination $\sum_{i=1,i\neq k}^{n} a_iE_i$. Then $E_k\sum_{i=1,i\neq k}^{n} a_iE_i = 0$ and $E_k^2 \neq E_k$, a contradiction.

Finally, it is necessary to find η_{ij} or ξ_{ij} for $i, j \leq n$. We find η_{ij}. Note that $\omega^{(k-1)r}$ occurs as an entry in E_k for $0 < k \leq n$ when $j - i = r \mod n$, or at the same entries as A_{r-1} is non-zero. Thus, η_{ij} will equal $\frac{\omega^{(i-1)(j-1)}}{n}$. Since $\sqrt{f_i} = 1$ for all i, $\psi_{ji} = \frac{n}{1}\overline{\frac{\omega^{(i-1)(j-1)}}{n}} = \omega^{n-(i-1)(j-1)}$. These entries give the character table of the group $\mathbb{Z}/_n$, which consists only of roots of unity and have first row and column with entries equal to 1.

The general case for abelian groups follows from the fact that any abelian group is a product of cyclic abelian groups and theorem 6.6 in [10, §7.2], which states that if $\phi : Q \times Q \to \mathbb{C}$ is a basic character of Q and $\theta : P \to Q$ a projection, then define $\theta'' : P \times P \to Q \times Q$ the diagonal map to take (p, p) to (p^θ, p^θ) then $\phi\theta : P^2 \to \mathbb{C}$ is a basic character of P [10, §7.2]. Then it is possible to work backwards from the the character tables of A and B to that of $A \times B$ using the projection maps from $A \times B$. □

These character tables possess many of the standard properties of ordinary group character tables, as mentioned earlier, including orthogonality relations almost identical to those in the group case [10, Thm. 6.6].

6.2. Character tables of hedral loops

Remark 6.13. The quotient of a hedral loop H by the normal subloop $\{\pm 1\}$ is always $\mathbb{Z}/_2 \times \mathbb{Z}/_2$, so four basic characters are necessarily induced from the characters of $\mathbb{Z}/_2 \times \mathbb{Z}/_2$ by Theorem 7.2 in [10]. There are at least 5 characters in total, since $(1, 1)^g \neq (1, -1)$ for all g in the multiplication group of Q and thus $(1, 1)$ and $(1, -1)$ are in distinct conjugacy classes in the hedral loop H. However, x and $-x$ are identified in $\mathbb{Z}/_2 \times \mathbb{Z}/_2$. Thus, in the surjective image of H in $\mathbb{Z}/_2 \times \mathbb{Z}/_2$, the conjugacy class containing $(1, x)$ and that containing $(1, -x)$ are always identified, but if $x \neq \pm y$, $(1, x)$ is not identified with $(1, y)$, so these conjugacy classes are always distinct for H since they are distinct in its image in $\mathbb{Z}/_2 \times \mathbb{Z}/_2$. Since every conjugacy class contains at least one element of the form $(1, x)$ any hedral loop with more than five conjugacy classes must have $(1, x)$, $(1, -x)$ in distinct conjugacy classes under the action of the multiplication group for at least one $x \in H$, $x \neq \pm 1$. Equivalently, they must have x and $-x$ in distinct conjugacy classes under the action of the stabilizer of 1 in the multiplication group. If a hedral loop has exactly 5 conjugacy classes, then it has the character table of Q_8 since four of its basic characters are induced from $\mathbb{Z}/_2 \times \mathbb{Z}/_2$, and the fifth basic character follows immediately from the orthogonality relations.

Lemma 6.14. *If H is commutative, then it is flexible (satisfying the identity $(xy)x = x(yx)$).*

Proof. Suppose it is not flexible but is commutative. Let x and y be two distinct positive non-identity elements of H. Then $(xy)x \neq x(yx)$ and necessarily $(xy)x = -x(yx)$. But then $(xy)x = -x(yx) = -(xy)x$ by commutativity, which is impossible. □

Lemma 6.15. *If H is noncommutative it has the character table of Q_8*

Proof. Prove the contrapositive. Suppose that H does not have the character table of Q_8. Let x and y be two distinct positive nonidentity elements in H. Let z be the other positive non-identity element.

Then there exists $x \in H^+ \backslash \{1\}$ such that $(1, x)^N \neq (1, -x)$, for all N in the multiplication group. In particular, $(yx)y^{-1}, y(xy^{-1}) \neq -x$. By the definition of hedral loops, this means that $(yx)y^{-1} = y(xy^{-1}) = x$, and thus $(yx)y = y(xy)$, since $y = \pm y^{-1}$. Additionally, $(xy^{-1})y = x$ and $x(y^{-1}y) = x$ for the same reason. Therefore, $xy^2 = (xy)y$ and symmetrically $y^2x = y(yx)$. Thus, $y(xy) = y(yx)$, $xy = yx$ and x is in the commutative center of the loop. Therefore, x associates with pairs of y and commutes with each y. Now, $x = (xy)y^{-1}$ and $x = y^{-1}(xy)$ and thus y^{-1} commutes with $xy = \pm z$ and hence y also commutes with $\pm z$, in addition to $\pm x$. Thus, y is also in the commutative center. Finally, z commutes with x and y since x and y are in the commutative center. Thus, the loop is commutative. By Lemma 6.14, H is also flexible. □

We now prove by cases that only hedral loops which are also abelian groups have character tables different from Q_8.

Lemma 6.16. *If H is commutative, flexible and $x^2 = 1$ for all $x \in H$ then H is an Abelian group or has character table identical to that of Q_8.*

Proof. For all $x, y \in H^+ \setminus \{1\}$, suppose $xy = yx = z$. Then H is isomorphic to $(\mathbb{Z}/_2)^3$. If instead, for all $x, y \in H^+ \setminus \{1\}$, $xy = yx = -z$. then it is isotopic and hence isomorphic to the former group by Lemma 2.3, using the isotopy (f, f, h) where $1^f = -1$, $(-1)^f = 1$ and $x^f = x$ for $x \neq \pm 1$ while $x^h = -x$ for all $x \neq \pm 1$ and $1^h = 1$, $(-1)^h = -1$. The former loop is isomorphic to the group $(\mathbb{Z}/_2)^3$.

Now, consider the other possible tables satisfying the required conditions. Note that $x = x^{-1}$ for all $x \in H$. Given distinct, positive, nonidentity elements x, y and z, each of the pairs $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ have products equal to a positive or negative element.

Suppose one pair has a negative product. Without loss of generality, choose $\{x, y\}$. Then $xyx = -zx = -y$, $yxy = -zy = -x$ and $xzx = yx = -z$, and thus $(1, w)$ and $(1, -w)$ are in the same conjugacy classes for $w = x, y, z$. Thus, by Remark 6.13, such loops have the character table of Q_8.

If two of the three pairs have a negative product, say $\{x, y\}$ and $\{x, z\}$, then $(zy)z = xz = -y$, $(zx)z = -yz = -x$ and $(yz)z = xz = -y$, and thus $(1, w)$ and $(1, -w)$ are in the same conjugacy classes for $w = x, y, z$. Therefore, by Remark 6.13, such loops have the character table of Q_8. □

Lemma 6.17. *If H is commutative, flexible and $x^2 = 1$ for all but one $x \in H^+ \setminus \{1\}$ then H has the character table of Q_8.*

Proof. Consider the possible multiplication tables satisfying the required conditions. For $x, y, z \in H^+$, let $x^2 = -1$. Given distinct, positive, nonidentity elements x, y and z, each of the pairs $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ have products equal to a positive or negative element. Two of these pairs contain an element squaring to -1, while all three contain an element squaring to 1.

Suppose first that for all $a, b, c \in H^+ \setminus \{1\}$ distinct, $ab = ba = c$. Then $-xyx = -y$, $-xzx = -z$, and $((1x)y)z = z^2 = 1$ while $((x^2)y)z) = -yz = -x$ and the character table of loops of this type is the same as that of Q_8.

Suppose instead that for all $a, b, c \in H^+ \setminus \{1\}$ distinct, $ab = ba = -c$. Then $(-xy)x = -y$, $(-xz)x = -z$, and $((1y)z)x = -x^2 = 1$ while $((xy)z)x) = -z^2x = -x$ and the character table of loops of this type is the same as that of Q_8.

Note that there are two distinct types of pairs of positive nonidentity elements. Namely, the pair $\{y, z\}$ in which both elements square to 1 and the pairs $\{x, y\}$ and

$\{x, z\}$ where one of the elements squares to -1, and it is necessary to check more possibilities than in the preceding result.

Suppose that $xy = -z$, $xz = y$ and $yz = x$. Then $yxy = -zy = -x$, $yzy = xy = -z$ and $(x(zy))y = x^2y = -y$ while $(x(z1))y = y^2 = 1$ and such a loop has five conjugacy classes and the same table as Q_8.

Suppose that $xy = -z$, $xz = -y$ and $yz = x$. Then $yxy = -yz = -x$, $yzy = yx = -z$ and $zyz = zx = -y$. Thus, such a loop has five conjugacy classes and the same character table as Q_8.

If $xy = z$, $xz = y$ and $yz = -x$ then $zyz = -zx = -y$, $zxz = zy = -x$ and $yzy = -yx = -z$. Thus, such a loop has five conjugacy classes and the same character table as Q_8.

If $xy = -z$, $xz = y$ and $yz = -x$, then $zxz = yz = -x$, $yzy = -xz = -z$ and $zyz = -xz = -y$. Thus, such a loop has five conjugacy classes and the same table as Q_8. □

Lemma 6.18. *If H is commutative, flexible and $x^2 = -1$ for all but one $x \in H^+$ then H has the character table of Q_8 or is isomorphic to the Abelian group $\mathbb{Z}/_2 \times \mathbb{Z}/_4$.*

Proof. Let $x^2 = y^2 = -1$, $z^2 = 1$. As in the preceding result there are two different types of pairs of positive non-identity elements: $\{x, y\}$, in which both x and y square to -1 and $\{x, z\}$ and $\{y, z\}$, in which only one of the elements squares to -1.

In the case when all $a, b, c \in H^+ \setminus \{1\}$ distinct satisfy $ab = ba = c$, $-yxy = -zy = -x$, $-xyx = -zx = -y$ and $-xzx = -yx = -z$ and loops of this type have five conjugacy classes and the character table of Q_8.

In the case when $ab = ba = -c$ for all $a, b, c \in H^+ \setminus \{1\}$ distinct, then $-yxy = zy = -x$, $-xyx = zx = -y$ and $-xzx = -yx = -z$ (just as in the preceding case) and loops of this type have five conjugacy classes and the character table of Q_8.

Now, suppose that $xy = -z$, $yz = x$ and $xz = y$. Then $(xy)z = -zz = -1$, $x(yz) = x^2 = -1$ and $(yx)z = -z^2 = -1$, $y(xz) = y^2 = -1$ and $(xz)y = y^2 = -1$, $x(zy) = x^2 = -1$ and $(zx)y = y^2 = -1$, $z(xy) = -z^2 = -1$ and $(yz)x = x^2 = -1$, $y(zx) = y^2 = -1$ and $(zy)x = x^2 = -1$, $z(yx) = -z^2 = -1$ and this loop is commutative and associative, hence an abelian group. Since it has elements of order 4 but no elements of order 8, it is isomorphic to $\mathbb{Z}/_4 \times \mathbb{Z}/_2$.

If $xy = -z$, $xz = -y$ and $yz = x$, then $-xyx = zx = -y$, $-xzx = yx = -z$ $zxz = -yz = -x$ and loops of this type have five conjugacy classes and the character table of Q_8.

If $xy = z$, $xz = -y$ and $yz = x$, then $zxz = -yz = -x$, $zyz = xz = -y$ and $-yzy = -xy = -z$ and loops of this type have five conjugacy classes and the character table of Q_8.

If $xy = z$, $xz = -y$ and $yz = -x$, then $(xy)z = zz = 1$, $x(yz) = -x^2 = 1$ and $(yx)z = zz = 1$, $y(xz) = -y^2 = 1$ and $(xz)y = -y^2 = 1$, $x(zy) = -x^2 = 1$ and $(zx)y = -y^2 = 1$, $z(xy) = z^2 = 1$ and $(yz)x = -x^2 = 1$, $y(zx) = -y^2 = 1$ and $(zy)x = -x^2 = 1$, $z(yx) = z^2 = 1$. Thus, the given quasigroup is associative, commutative, has an element of order 4 and has no element of order 8 and is thus isomorphic to $\mathbb{Z}/_2 \times \mathbb{Z}/_4$. □

Lemma 6.19. *If H is commutative, flexible and $x^2 = -1$ for all $x \in H^+ \setminus \{1\}$ then H has the character table of Q_8.*

Proof. Suppose first that for all $a, b, c \in H^+ \setminus \{1\}$ distinct, $ab = ba = c$. Then, $-xyx = -zx = -y$, $-xzx = -yx = -z$ and $-yxy = -zy = -x$ and loops of this type have five conjugacy classes and the character table of Q_8.

Suppose instead that for all $a, b, c \in H^+ \setminus \{1\}$ distinct, $ab = ba = -c$. Then, $-xyx = zx = -y$, $-xzx = yx = -z$ and $-yxy = zy = -x$, as in the previous case, and loops of this type have five conjugacy classes and the character table of Q_8.

Suppose $xy = -z$, $xz = y$ and $yz = x$. Then $-zyz = -xz = -y$, $-zxz = -yz = -x$ and $(-y(xz))(-z) = -y^2(-z) = -z$, while $(-y(x1))(-z) = --z(-z) = 1$ and loops of this type have five conjugacy classes and the character table of Q_8.

Suppose $xy = -z$, $xz = -y$ and $yz = x$. Then $-xyx = zx = -y$, $-xzx = yx = -z$ and $-x((xy)z) = -x((-z)z) = -x$ while $-x((1y)z) = -x^2 = 1$. Thus, loops of this type have the character table of Q_8. □

This exhausts all possible commutative hedral loops, and thus all hedral loops, since noncommutative hedral loops have the character table of Q_8 by Lemma 6.15.

The preceding results give the following theorem.

Theorem 6.20. *Every hedral loop has the character table of Q_8 or of an abelian group.*

6.2. Subloops

Every hedral loop possesses $\{\pm 1\}$ as a normal subgroup. Furthermore, any other element generates a subloop isomorphic to either $\mathbb{Z}/_2$ if it squares to 1 or $\mathbb{Z}/_4$ if it squares to -1 If $x^2 = 1$, then x and $-x$ generate a subloop isomorphic to $\mathbb{Z}/_2 \times \mathbb{Z}/_2$. Since ± 1, $x \neq \pm 1$ and $y \neq \pm x$ generate the whole loop, this exhausts all subloops. The question is which are normal. Those which are of order 4 are necessarily normal by Lemma 2.7

A loop of order 2 generated by x squaring to 1 is normal if and only if the stabilizer of 1 in the multiplication group also stabilizes x. Such a loop would necessarily have at least 6 conjugacy classes. Thus, by Theorem 6.20, there is no such loop that is not an abelian group.

The above results are summarized in the following theorem.

Theorem 6.21. *If H is a hedral loop but not an abelian group, its only normal subgroup of order 2 is generated by -1. All its other subloops are isomorphic to $\mathbb{Z}/_2$ or are normal subgroups of order 4 isomorphic to $\mathbb{Z}/_2 \times \mathbb{Z}/_2$ or $\mathbb{Z}/_4$.*

7. The octonion loop

The octonion loop is a standard signed superquasigroup, and appears as an extension of $(\mathbb{Z}/_2)^3$ by $\mathbb{Z}/_2$. Since the octonions themselves are alternative, O_{16} is Moufang [4]. One possible representation of the multiplication table of the transversal of this loop, which we label O_{16}, is given in Table 1.

It would be nice if this loop could be produced from the basic signed superquasigroups in some way, as both C_4 (the group of units of the Gaussian integers) and Q_8 (the group of units of quaternions with integral coefficients) can be so constructed. This would complete the series of composition algebras. However, this is impossible by Corollary 5.8. The next best thing would be to produce it as a signed product of some other

1	i_0	i_1	i_2	i_3	i_4	i_5	i_6
i_0	-1	i_2	i_3	i_4	i_5	i_6	i_1
i_1	$-i_2$	-1	i_4	i_5	i_6	i_0	i_3
i_2	$-i_3$	i_4	-1	i_6	i_0	i_1	i_5
i_3	$-i_4$	$-i_5$	$-i_6$	-1	i_1	i_2	i_0
i_4	i_5	$-i_6$	$-i_0$	$-i_1$	-1	i_3	i_2
i_5	$-i_6$	$-i_0$	$-i_1$	$-i_2$	$-i_3$	-1	i_4
i_6	$-i_1$	$-i_3$	$-i_5$	$-i_0$	$-i_2$	$-i_4$	-1

Table 1: A multiplication table of the positive elements of the octonion loop [3, pg. 65].

standard signed superquasigroups or signed superloops. In this section, this is shown to be impossible as well.

Lemma 7.1. *Any loop isotopic to the octonion loop is isomorphic to the octonion loop.*

Proof. The octonion loop has two sided inverses fulfilling $(xy)y^{-1} = x$, $y^{-1}(yx) = x$ (or equivalently, satisfying $L(x)^{-1} = L(x^{-1})$, $R(x)^{-1} = R(x^{-1})$. That is, it is an inverse property loop [2, §2], which follows immediately from its being a Moufang loop. Any principal isotopy to O_{16} will be of the form $g^\alpha h^\beta = (gR(1^\beta)^{-1})(hL(1^\alpha)^{-1}) = g * h$. by Lemma 6.4. By the inverse property, this means that $g^\alpha h^\beta = (g(1^\beta)^{-1})((1^\alpha)^{-1}h)$. In particular $1^\beta 1^\alpha = 1$ and thus $1^\alpha = (1^\beta)^{-1}$. If $1^\alpha = x$ then $1^\beta = -x$, assuming $x \neq \pm 1$. If $x = \pm 1$, the resulting isotopy is trivial. If not, the fact that O_{16} is Moufang and $xy = -yx$, for $x \neq \pm y$, means that any principle isotopy with another loop leaves the multiplication operation unchanged. □

Lemma 7.2. *Let Q be a hedral loop and let C be a signed supergroup of order 4 with $\mathbb{Z}/_2$-grading. Then $Q \widehat{\times} C$ will never be isotopic to the octonion loop.*

Proof. Since all loop isotopies with O_{16} are isomorphisms, it is sufficient to consider isomorphisms. This superproduct is already a loop. Assume for the sake of contradiction that $Q \widehat{\times} C$ is isomorphic to O_{16}. Let the elements of Q be ± 1, $\pm x$, $\pm y$ and $\pm z$. Now, the subset $Q \widehat{\times} 1$ of $Q \widehat{\times} C$ will be a subloop isomorphic to Q by Lemma 5.3. Thus the subloop will have 8 elements. But the only 8 element subloops of the octonion loop are quaternion groups, so Q is the quaternion group. But the superproduct of two supergroups is again a supergroup, therefore O_{16} is a supergroup, a contradiction. □

Lemma 7.3. *The signed superproduct of two standard signed superquasigroups A and B is a loop if and only if A and B are loops.*

Proof. Let A and B be standard signed superquasigroups. If A is not a loop, then there exists $a \in A$ such that $a1 = -a$ or $1a = -a$. Then either $(a\widehat{\times}1)(1\widehat{\times}1) = -(a\widehat{\times}1)$ or $(1\widehat{\times}1)(a\widehat{\times}1) = -(a\widehat{\times}1)$ and $A\widehat{\times}B$ is not a loop. The same reasoning holds if B is not a loop. If A and B are loops, then $A\widehat{\times}B$ is always a loop by Proposition 3.14. □

Proposition 7.4. *Let A and B be signed superloops with non-trivial grading. Then $A\widehat{\times}B$ is not isomorphic, hence not isotopic, to the octonion loop.*

Proof. Suppose that such A and B do exist. The signed superloop $A\widehat{\times}B$ is isotopic to O_{16} if and only if it is isomorphic to O_{16} by Lemma 7.1. The product of the order of A with that of B must be 32. If one of the two superquasigroups has order 2, it is the unit basic superquasigroup, while if one of the two has order 1, it is the unit superquasigroup, which is not signed. In both cases it is trivially graded, so it is excluded from consideration by assumption. Therefore, neither A nor B has order 32, 16, 2 or 1.

Thus one of the loops A and B has order 8 and the other order 4. Assume without loss of generality that A is of order 8 by Proposition 3.15, as the octonion loop is isomorphic to its opposite and A is hedral if and only if A^{op} is hedral. Then B is of order 4 and all signed superloops of order 4 are signed supergroups (in fact, all loops of order 4 are groups) so B is isomorphic to either $\mathbb{Z}/4$ or $(\mathbb{Z}/_2)^2$.

If the transversal of the product of A and B is taken, every positive non-identity element of $A\widehat{\times}B$ anticommutes with every element other than itself, and the diagonal elements of the multiplication table are all -1. Thus, $(a\otimes b)(a\otimes b) = (-1)^{|b|\,|a|}a^2\otimes b^2 = -1\otimes 1$. Now $b^2 = \pm 1$, so, $a^2 = \pm 1$. For $a \neq a' \in A$ both positive non-identity, $(a\otimes 1)(a'\otimes 1) = -(a'\otimes 1)(a\otimes 1)$ and $aa' = -a'a$. The third positive non-identity element, a'', also anticommutes with a and a' for the same reasons.

Now, let $aa' = x$. Since A is a signed superloop $a1 = a$, $1a' = a'$, $a(-1) = -a$ and $-1a' = a'$. Note that x cannot equal $\pm a$ or $\pm a'$ without contradicting the definition of a quasigroup, and since $a^2 = a'^2 = \pm 1$, x cannot be ± 1 for the same reason. Thus x is equal to $\pm a''$ and this is sufficient to show that the transversal of the multiplication table of A, ignoring signs, is the same as the multiplication table of $\mathbb{Z}/_2\times\mathbb{Z}/_2$. For B, there is only one positive non-identity element, so the transversal of the multiplication table without regard for signs must be that of $\mathbb{Z}/_2\times\mathbb{Z}/_2$.

Thus A is a hedral loop, B is a group of order 4, and it was shown in Proposition 7.4 that this is not a signed superproduct of A and B. □

Corollary 7.5. *No superproduct of signed superloops is isotopic to the octonion loop.*

Proof. The case when grading is nontrivial is dealt with by Proposition 7.4 and Proposition 7.2, so it is only necessary to deal with the case when the grading on one or both of A and B is trivial. The reasoning used in Proposition7.4 gives that A is order 8 and B is order 4, or vice versa. As the signed superproduct is symmetric when grading is trivial, the order does not matter. By Lemma 7.1, it is sufficient to consider isomorphism. Elements in $(A,1)$ commute with elements in $(1,B)$, and since B is a loop of order 4, it is an abelian group. Then $(a,b')(1,b) = (a,b'b) = (1,b)(a,b)$ and the center of this signed product is at least order 4, which means the signed product cannot be isomorphic to the octonion loop, which has center $\{\pm 1\}$. □

Corollary 7.6. *No superproduct of standard signed superquasigroups is isotopic to the octonion loop.*

Proof. Any product $A\widehat{\times}B$ of standard superquasigroups is isotopic to a product of signed superloops by Theorem 5.7, and the preceding corollary gives that such a superproduct is not isotopic to the octonion loop. □

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