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On the symplectic groups $\mathbf{PSp(8,q)}$ by a conjugacy class size and order of the group

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Abstract. We prove that symplectic group $PSp(8,q)$, where $q=2^n$, $p=q^4+1$ is a prime number can be uniquely determined by its order and one conjugacy class size.

1. Introduction

Let G be a finite group. This paper focuses on characterizing groups by their order and conjugacy class sizes. For this purpose, the set of all conjugacy class sizes of a group G is denoted by $N(G)$. Also, we denote the conjugacy class of size of prime number p by $\frac{|PSp(8,q)|}{p}$. For every integer n denote by $\pi(n)$ the set of all prime divisors of n. The prime graph $\pi(G)$ of G is constructed upon the vertex set $\pi(|G|)$ in such a way that two distinct primes u and v are joined by an edge if and only if G has an element of order uv.

Let $t(G)$ be the number of connected components of $\pi(G)$. These components will be denoted by $\pi_1, \pi_2, \ldots, \pi_{t(G)}$. If G is of even order, then π_1 is chosen to be the component in which 2 is a vertex. We denote m_1, m_2,...,$m_t(G)$ to be the integers such that $|G| = m_1, \ldots, m_t(G)$ and $\pi(m_i)$ is the vertex set of π_i. If m_i is odd, call π_i an odd order component [9]. The starting point for our discussion is from a conjecture of J. G. Thompson, which is Problem 12.38 in the Kourovka notebook [16] is as follows:

Thompson's conjecture. *Let G be a group with trivial center. If M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.*

Next, for example the authors in ([2, 3, 4, 5, 7, 8, 10, 11]), proved that the sporadic simple groups, alternating group Alt_{10}, $PSU_3(q)$, projective special linear group $PSL(4,4)$ and projective special linear group $PSL(2,p)$, $PSL(n,2)$, orthogonal group ${}^2D_n(2)$,${}^2D_{n+1}(2)$, $PSp(2n,2)$, alternating group Alt_n where $n \in \{p, p+1, p+2\}$ and symmetric group

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$Sym(p)$ where p is a prime number and $PSL(5,q)$ and Ree group ${}^2G_2(q)$, where $q \pm \sqrt{3q}+1$ is a prime number are characterized by using the order of the group and the conjugacy class of size. In this paper, we prove that symplectic groups $PSp(8,q)$, where $q=2^n$, $p=q^4+1$ is a prime number can be uniquely determined by its order and one conjugacy class of size. In fact, we prove the following main theorem.

Main Theorem. *Let G be a group with $|G|=|PSp(8,q)|$, where $q=2^n$. If $p=q^4+1$ is a prime, then $G\cong PSp(8,q)$ if and only if G has a conjugacy class of size $CS(G)=\frac{|PSp(8,q)|}{p}$.*

2. Notation and preliminaries

Lemma 2.1. [13] *Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G)=2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K|-1$;*
3. *K is nilpotent.*

Definition 2.2. The group G is called a 2-*Frobenius group* if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3. [6] *Let G be a 2-Frobenius group of even order. Then*

1. *$t(G)=2$, $\pi(H)\cup\pi(G/K)=\pi_1$ and $\pi(K/H)=\pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 2.4. [18] *Let G be a finite group with $t(G)\geqslant 2$. Then one of the following statements holds:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group;*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Next, we summarize the relevant information in Tables – below where we use them in proof of main theorem.

Table 1. The order components of simple groups with $t(G) = 2$, p is an odd prime number. ([14])

Group	π_1	π_2
$Alt(n)$, $n \neq 5, 6$, $n = p, p+1, p+2$, n and $n-2$ not both prime	$\{2, 3, \ldots, q\}$, $q < n-2$	p
$A_{p-1}(q)$, $(p, q) \neq (3, 2), (3, 4)$	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$, $q - 1 \mid p + 1$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - 1)$	$\frac{q^p-1}{(q-1)}$
${}^2A_{p-1}(q)$	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$
${}^2A_p(q)$, $q + 1 \mid p + 1$, $(p, q) \neq (3, 3), (5, 2)$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - (-1)^i)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$	$2^6.3^4$	5
$B_n(q)$, $n = 2^m \geqslant 4$, q odd	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{2}$
$B_p(3)$	$3^{p^2}(3^p + 1) \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p-1}{2}$
$C_n(q)$, $n = 2^m \geqslant 2$	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$, $q = 2, 3$	$q^{p^2}(q^p + 1) \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$, $p \geqslant 5$, $q = 2, 3, 5$	$q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$, $q = 2, 3$	$\frac{1}{(2,q-1)} q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{q-1}$
${}^2D_n(q)$, $n = 2^m \geqslant 4$	$q^{n(n-1)} \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$, $n = 2^m + 1 \geqslant 5$	$2^{n(n-1)}(2^n + 1)(2^{n-1} - 1) \prod_{i=1}^{n-2}(2^{2i} - 1)$	$2^{n-1} + 1$
${}^2D_p(3)$, $p \neq 2^m + 1$, $p \geqslant 5$	$3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$, $n \neq 2^m+1 \neq p$, $m \geqslant 2$	$\frac{1}{2} 3^{n(n-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p+1}{4}$
$G_2(q)$, $q \equiv \alpha(\text{mod} 3)$, $\alpha = \pm 1$, $q > 2$	$q^6(q^3 - \alpha)(q^2 - 1)(q + \alpha)$	$q^2 - \alpha q + 1$
${}^3D_4(q)$	$q^{12}(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1)$	$q^4 - q^2 + 1$
$F_4(q)$, q odd	$q^{24}(q^8 - 1)(q^6 - 1)^2(q^4 - 1)$	$q^4 - q^2 + 1$
$E_6(q)$	$q^{36}(q^{12}-1)(q^8-11)(q^6-1)(q^5-1)(q^3-1)(q^2-1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
${}^2E_6(q)$, $q > 2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)(q^5+1)(q^3+1)(q^2-1)$	$\frac{q^6-q^3+1}{(3,q+1)}$
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2$	13

Table 2. Components of the sporadic groups. ([14])

Group	π_1	π_2	π_3	π_4	π_5	π_6
M_{11}	$2^4{\cdot}3^2$	5	11			
M_{12}	$2^6{\cdot}3^3{\cdot}5$	11				
M_{22}	$2^7{\cdot}3^2$	5	7	11		
M_{23}	$2^7{\cdot}3^2{\cdot}5{\cdot}7$	11	23			
M_{24}	$2^{10}{\cdot}3^3{\cdot}5{\cdot}7$	11	23			
J_1	$2^3{\cdot}3{\cdot}5$	7	11	19		
J_2	$2^7{\cdot}3^3{\cdot}5^2$	7				
J_3	$2^7{\cdot}3^5{\cdot}5$	17	19			
J_4	$2^{21}{\cdot}3^3{\cdot}5{\cdot}7{\cdot}11^3$	23	29	31	37	43
HS	$2^9{\cdot}3^2{\cdot}5^3$	7	11			
Ru	$2^{14}{\cdot}3^3{\cdot}5^3{\cdot}7{\cdot}13$	29				
Sz	$2^{13}{\cdot}3^7{\cdot}5^2{\cdot}7$	11	13			
He	$2^{10}{\cdot}3^3{\cdot}5^2{\cdot}7^3$	17				
ON	$2^9{\cdot}3^4{\cdot}5{\cdot}7^3$	11	19	31		
Mcl	$2^7{\cdot}3^6{\cdot}5^3{\cdot}7$	11				
Ly	$2^8{\cdot}3^7{\cdot}5^6{\cdot}7{\cdot}11$	31	37	67		
Co_1	$2^{21}{\cdot}3^9{\cdot}5^4{\cdot}7^2{\cdot}11{\cdot}13$	23				
Co_2	$2^{18}{\cdot}3^6{\cdot}5^3{\cdot}7$	11	23			
Co_3	$2^{10}{\cdot}3^7{\cdot}5^3{\cdot}7{\cdot}11$	23				
F_{22}	$2^{17}{\cdot}3^9{\cdot}5^2{\cdot}7{\cdot}11$	13				
F_{23}	$2^{18}{\cdot}3^{13}{\cdot}5^2{\cdot}7{\cdot}11{\cdot}13$	17	23			
F'_{24}	$2^{21}{\cdot}3^{16}{\cdot}5^2{\cdot}7^3{\cdot}11{\cdot}13$	17	23	29		
$F_1 = M$	$2^{46}{\cdot}3^{20}{\cdot}5^9{\cdot}7^6{\cdot}11^2{\cdot}13^3{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}41$	47	59	71		
$F_2 = B$	$2^{41}{\cdot}3^{13}{\cdot}5^6{\cdot}7^2{\cdot}11{\cdot}13{\cdot}17{\cdot}19{\cdot}23$	31	47			
$F_3 = Th$	$2^{15}{\cdot}3^{10}{\cdot}5^3{\cdot}7^2{\cdot}13$	19	31			
$F_5 = Ha$	$2^{14}{\cdot}3^6{\cdot}5^6{\cdot}7{\cdot}13$	19				

Lemma 2.5. [19] *Let q, k, l be natural numbers. Then*

1. $(q^k - 1, q^l - 1) = q^{(k,l)} - 1.$
2. $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
3. $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geqslant 2$ and $k \geqslant 1$ the inequality $(q^k - 1, q^k + 1) \leqslant 2$ holds.

3. Proof of the Main Theorem

Note that this paper focuses on the symplectic group $PSp(8, q)$. For this purpose, we denote the symplectic groups $PSp(8, q)$ and prime number $q^4 + 1$ by M and p respectively. Furthermore by [15], $PSp(8, q)$ has conjugacy class of size $CS(G) = \frac{|PSp(8,q)|}{p}$. It suffices to prove that if $CS(G) = CS(PSp(8, q)$ and $|G| = |PSp(8, q)|$ then $G \cong PSp(8, q)$. By the assumption on q, there exists an element α of order p in G such that $C_G(\alpha) =< \alpha >$ and $C_G(\alpha)$ is a Sylow p-subgroup of G. By the Sylow's theorem, we have that $C_G(\beta) =< \beta >$ for any element β in G of order p. In the following we prove p is an isolated vertex in $\Gamma(G)$. We note that $|PSp(8, q)| = \frac{q^{16}(q^2-1)(q^4-1)(q^6-1)(q^2-8)}{2}$ (q is odd) and $CS(PSp(8, q)) = \frac{|PSp(8,q)|}{p}$.

Lemma 3.1. *p is an isolated vertex in $\Gamma(G)$.*

Proof. We shall prove that p is an isolated vertex of Γ(G). Suppose for contradiction, that there exists $t \in \pi(G) \setminus \{p\}$ such that $tp \in \pi_e(G)$. So $tp \geqslant 2p = 2(q^4 + 1) > q^4 + q$, thus $k(G) > q^4 + q$. As a result $t(G) \geqslant 2$. □

So by Lemma 2.4 we have

Lemma 3.2. *The group G is neither a Frobenius group and a 2-Frobenius group.*

Lemma 3.3. *The group G is isomorphic to the group M.*

Proof. By Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. At the moment by Lemma 3.2 and Lemma 2.3 implies that G is neither a Frobenius group and a 2-Frobenius group. Thus only the case 3 of Lemma 2.4 occur. Let G have a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, H is a nilpotent group, and $|G/K|$ divides $|Out(K/H)|$. Since K/H is a non-abelian simple group and p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. Now according to the classification of the finite simple groups we know that the possibilities for K/H are an alternating group Alt_m, $m \geqslant 5$, 26 sporadic groups, and a simple group of Lie types. First, we suppose that K/H is isomorphic to an alternating group.

Step 1. Suppose that $K/H \cong Alt_m$, $m \geqslant 5$. On the other hand, by [18], $\pi(Alt_m) = r', r' - 2$. We know that $|Alt_m|$ divide $|G|$. First, assume $q^4 + 1 = r'$ now since that $q \geqslant 2$ so we deduce $p \geqslant 17$. If $p = r = 17$ then $|Alt_m| \nmid |G|$, which is a contradiction. Next, if $q^4 + 1 = r' - 2$ then $r' = q^4 + 3$. Now since $|Alt_m| \nmid |G|$, which is a contradiction.

Step 2. Suppose that K/H is isomorphic to sporadic group. On the other hand, by[18], $\pi(S) = \{11, 13, 17, 19, 23, 29, 31, 43, 47, 67, 71\}$. Now, for example we consider $q^4 + 1 = 71$, so $q^4 = 70$ which is a contradiction.

Step 3. Here, we consider Lie type groups and the following isomorphism.

(i). Suppose that $K/H \cong {}^3D_4(q')$, $q \equiv \pm 2 (\text{mod } 5)$. On the other hand, by [18],

$\pi({}^3D_4(q')) = q'^4 - q'^2 + 1$. We know that $|{}^3D_4(q')|$ divided $|G|$, so $q'^{12}(q'^8 + q'^4 + 1)(q'^6 - 1)(q'^2 - 1) \mid \frac{q^3(q^3-1)(q^2-1)}{(3,q-1)}$. Now, we consider $q^4 + 1 = q'^4 - q'^2 + 1$, so $q^4 = q'^4 - q'^2$. Since that $|{}^3D_4(q')| \mid |G|$, so must be $q'^{12}(q'^8+q'^4+1)(q'^6-1)(q'^2-1) \mid (q'^4 - q'^2)^4(q'^4 - q'^2)^2 - 1)(q'^4 - q'^2)^{3/2} - 1)\ (q'^4 - q'^2)^{1/2} - 1)$, where this is a contradiction.

(II). Suppose that $K/H \cong E_6(q')$. On the other hand, by [18], $\pi(E_6(q')) = \frac{q'^6+q'^3+1}{(3,q'-1)}$. First, assume $(3, q' - 1) = 1$ then $q^4 + 1 = q'^6 + q'^3 + 1$. It follows that $q^4 = q'^6 + q'^3$. On the other hand, $|E_6(q')| \mid |G|$, as $q'^{36}(q'^{12} - 1)(q'^9 - 1)(q'^8 - 1)(q'^6 - 1)(q'^5 - 1)(q'^2 - 1) \mid q^{16}(q^8 - 1)(q^6 - 1)(q^2 - 1)$. Now, since $q^4 = q'^6 + q'^3$ but $|E_6(q')| \nmid |G|$, which is a contradiction. Now, if $(3, q' - 1) = 3$ then $q^4 + 1 = \frac{q'^6+q'^3+1}{3}$. It follows that $3q^4 + 2 = q'^6 + q'^3$ but $|E_6(q')| \nmid |G|$, which is a contradiction.

(III). Suppose that $K/H \cong E_8(q')$. On the other hand, by [18], $\pi(E_8(q')) = \frac{q'^{10}\pm q'^5+1}{q'^2\pm q'+1}$, $q'^8 - q'^4 + 1$, $\frac{q'^{10}+1}{q'^2+1}$. First, we consider $q^4 + 1 = \frac{q'^{10}\pm q'^5+1}{q'^2\pm q'+1}$ so $q^4 = \frac{q'^{10}\pm q'^5+1}{q'^2\pm q'+1} - 1$. Now, since $|E_8(q')| \nmid |G|$, which is a contradiction. A similar argument applies if $K/H \ncong E_7(q')$.

(IV). Suppose that $K/H \cong F_4(q')$. On the other hand, by [18], $\pi(F_4(q')) = q'^4 + 1, q'^4 - q'^2 + 1$. First, if $q^4 + 1 = q'^4 + 1$ then $q = q'$. Now, since $|F_4(q')| \nmid |G|$, which is a contradiction. For $q^4 + 1 = q'^4 - q'^2 + 1$ we have a contradiction.

(V). Suppose that $K/H \cong {}^2F_4(q')$, $q' = 2^{2m'+1} \geqslant 2$. Also by [18], $\pi({}^2F_4(q')) = q'^2 \pm \sqrt{2q'^3} \pm q' \pm \sqrt{2q'} + 1$. On the other hand, we know $|{}^2F_4(q')|$ divided $|G|$. Now, we consider $p = p'$, as $q^4 + 1 = q'^2 \pm \sqrt{2q'^3} \pm q' \pm \sqrt{2q'} + 1$. Next, as $q^4 = q'^2 \pm \sqrt{2q'^3} \pm q' \pm \sqrt{2q'}$. Since that $|{}^2F_4(q')| \nmid |G|$, where this is a contradiction.

(VI). Suppose that $K/H \cong {}^2B_2(q')$, $q' = 2^{2m'+1} \geqslant 2$. Also by [18], $\pi({}^2B_2(q')) = q' \pm \sqrt{2q'} + 1$. On the other hand, we know $|{}^2B_4(q')|$ divided $|G|$. Now, we consider $p = p'$, as $q^4 + 1 = q' \pm \sqrt{2q'} + 1$. Next, as $2^{8n} = 2^{2n'+1} \pm 2^{n'+1}$. Since that $|{}^2B_2(q')| \nmid |G|$, where this is a contradiction.

(VII). Suppose that $K/H \cong G_2(q')$. Now, by [18], $\pi(G_2(q')) = q'^2 \pm q' + 1$. On the other hand, we know $|G_2(q')|$ divided $|G|$. Now, we consider $p = p'$, as $q^4 + 1 = q'^2 \pm q' + 1$. So $q^4 = q'^2 \pm q'$ Since that $|G_2(q')| \nmid |G|$, where this is a contradiction.

(VIII). Suppose that $K/H \cong {}^2A'_n(q')$, $n' \geqslant 1$. Also by [18], $\pi({}^2A_{n'}(q')) = \frac{q'^{n'}+1}{(q'+1)(n',q'+1)}$. On the other hand, we know $|{}^2A_{n'}(q')|$ divided $|G|$. Now, we consider $p = p'$, as $q^4 + 1 = \frac{q'^{n'}+1}{(q'+1)(n',q'+1)}$. Next, Since that $|{}^2A_{n'}(q')| \nmid |G|$, where this is a contradiction. For $K/H \ncong L_{n'+1}(q')$, $K/H \ncong D_{n'}(q')$, we have a contradiction, similarly.

Hence, $K/H \cong PSp(8,q')$. Now since that $|K/H| = |PSp(8,q')| = |G|$ and also $p \in \pi(K/H)$ so $p = p'$. So $q^4 + 1 = q'^4 + 1$. Thus $q = q'$. On the other hand, $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, thus $H = 1$, $G = K \cong PSp(8,q')$. $\square$

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