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Second semimodules over commutative semirings

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Abstract. Let R be a semiring. A non-zero subsemimodule S of an R-semimodule M is *second* if for each $a \in R$, we have $aS = S$ or $aS = 0$. The aim of this paper is to study the notion of second subsemimodules of semimodules over commutative semirings.

1. Introduction

A *semiring* is a non-empty set R together with two binary operations addition $(+)$ and multiplication $(\cdot)$ such that $(R, +)$ is a commutative monoid with identity element 0; $(R, .)$ is a monoid with identity element $1 \neq 0$; $0a = 0 = a0$ for all $a \in R$; $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every $a, b, c \in R$. R is called a *commutative semiring* if the monoid $(R, .)$ is commutative. In this paper we assume that all semirings are commutative.

A non-empty subset I of a semiring R is called an *ideal* of R if $a+b \in I$ and $ra \in I$ for all $a, b \in I$ and $r \in R$. An ideal I of a semiring R is *subtractive* if $a + b \in I$ and $b \in I$ imply that $a \in I$ for all $a, b \in R$. Let $(M, +)$ be an additive abelian monoid with additive identity 0_M. Then M is called an R-*semimodule* if there exists a scalar multiplication $R \times M \to M$ denoted by $(r, m) \mapsto rm$, such that $(rs')m = r(s'm)$; $r(m+m') = rm+rm'$; $(r+r')m = rm+r'm$; $1m = m$ and $r0_M = 0_M = 0m$ for all $r, r' \in R$ and all $m, m' \in M$. A *subsemimodule* N of a semimodule M is a non-empty subset of M such that $m + n \in N$ and $rn \in N$ for all $m, n \in N$ and $r \in R$. A subsemimodule N of an R-semimodule M is called a *subtractive subsemimodule* or a k-*subsemimodule* if $x, x+y \in N$ implies $y \in N$. It is clear that N is subtractive if and only $N = \overline{N}$, where $\overline{N} = \{r \in R | r + x = y \ for \ some \ x, y \in N\}$. For each subsemimodule N of M, $\overline{N}$ is a k-subsemimodule of M. M itself is a k-subsemimodule of M and 0 is also a k-subsemimodule of M.

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A proper subsemimodule N of an R-semimodule M is said to be *prime* in M, if $rx \in N$ with $r \in R$ and $x \in M$ implies $r \in (N :_R M)$ or $x \in N$ [9]. The concept of second submodule of an R-module (as a dual notion of prime submodules) was introduced and studied by S.Yassemi in 2001. A non-zero submodule N of an R-module M is called *second* if for each $a \in R$, we have $aN = N$ or $aN = 0$ [11]. This notion has obtained a great attention by many authors and now there is a considerable amount of research concerning this class of modules. For more information about this class of modules we refer the reader to [5]. The algebraic structure of semirings, that are considered as a generalization of rings, plays an important role in different branches of mathematics, especially in applied sciences and computer engineering. The purpose of this paper is to study second semimodules as a dual notion of prime semimodules and extend some of the results of [2, 3, 11] to semimodules over commutative semirings.

2. Second subsemimodules

Definition 2.1. We say that a non-zero subsemimodule S of an R-semimodule M is *second* if for each $a \in R$, we have $aS = S$ or $aS = 0$. Also, a non-zero R-semimodule M is *second*, if M is a second subsemimodule of M.

Proposition 2.2. *Let N be a subsemimodule of an R-semimodule M. Then the following are equivalent:*

- (a) *N is a second subsemimodule of M;*
- (b) *$N \neq 0$ and $aS \subseteq K$ implies that $aS = 0$ or $S \subseteq K$ for each $a \in R$ and subsemimodule K of M.*

Proof. $(a) \Rightarrow (b)$. This is clear.

$(b) \Rightarrow (a)$. This follows from the fact that $aN \subseteq aN$. □

Definition 2.3. Let M be an R-semimodule. We say that a subsemimodule N of M is a *minimal subsemimodule* of M if there is no subsemimodule K of M satisfying $0 \subset K \subset N$.

Example 2.4. Assume that $\mathbb{Z}_0^+$ is the set of non-negative integers and consider the $\mathbb{Z}_0^+$-semimodule $M = \mathbb{Z}_{16}$ and take $N = \{\bar{0}, \bar{8}\}$ as a subsemimodule of M. Then N is a minimal subsemimodule of M.

Remark 2.5. Clearly, every minimal subsemimodule of R-semimodule M is a second subsemimodule of M. But the converse is not true in general.

For example, consider the $\mathbb{Z}_0^+$-semimodule $M = \mathbb{Q}^+$. Then M is a second subsemimodule of M which is not a minimal subsemimodule of M.

An R-semimodule M is called a *comultiplication R-semimodule* if for any subsemimodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [10]. Clearly, $(0 :_M I)$ is a subtractive subsemimodule of M for each ideal I of R. Thus if M is a comultiplication R-semimodule, then every subsemimodule of M is a subtractive subsemimodule.

An R-semimodule M is called a k-comultiplication R-semimodule if for any subtractive subsemimodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$ [10].

Proposition 2.6. *Let R be a semiring, M an R-semimodule and N a subsemimodule of M. Then we have the following.*

(a) *If N is a second subsemimodule of M, then $Ann_R(N)$ is a prime k-ideal of R. In this case, we say that N is $Ann_R(N)$-second.*

(b) *If N is a subsemimodule of a comultiplication R-semimodule M such that $Ann_R(N)$ is prime ideal of R, then N is a second subsemimodule of M.*

Proof. (a). First note that by [7, Proposition 2.3], $Ann_R(N)$ is a k-ideal of R. Let N be a second subsemimodule of M. Then $N \neq 0$ and so $Ann_R(N) \neq R$. Now suppose that $rs \in Ann_R(N)$. Then $sN \subseteq (0 :_M r)$. Thus $sN = 0$ or $rN = 0$, as needed.

(b). Suppose that N is a subsemimodule of a comultiplication R-semimodule M such that $Ann_R(N)$ is prime ideal of R. Let $aN \subseteq K$ for some $a \in R$ and subsemimodule K of M. As M is a comultiplication R-semimodule, there exists an ideal I of R such that $K = (0 :_M I)$. Therefore, $aI \subseteq Ann_R(N)$. If $I \subseteq Ann_R(N)$, then $N \subseteq K$ and we are done. So suppose that $b \in I \setminus Ann_R(N)$. Then $ab \in Ann_R(N)$. This implies that $a \in Ann_R(N)$ or $b \in Ann_R(N)$. Since $b \notin Ann_R(N)$, we have $a \in Ann_R(N)$, as needed. □

The following example shows that the converse of Proposition 2.6 (a) is not true in general.

Example 2.7. Let R be $\mathbb{Z}^+ = \mathbb{Z}^+ \cup \{0\}$. Then $M = \mathbb{Z}^+ \times \mathbb{Z}^+$ is an R-semimodule. Consider the subsemimodule $N = 0 \times 4\mathbb{Z}^+$ of M. Then $Ann_R(N) = 0$ is prime ideal of R but N is not a second subsemimodule of M. Because $2N \neq N$ and $2N \neq 0$.

A proper ideal I of a semiring R is said to be a *strong ideal* if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$ [4].

Proposition 2.8. *Let M be a finitely generated comultiplication R-semimodule and P be a strong prime k-ideal of R containing $Ann_R(M)$. Then $(0 :_M P)$ is a second subsemimodule of M.*

Proof. By Proposition 2.6 (b), it is enough to show that $Ann_R((0 :_M P)) = P$. Let $r(0 :_M P) = 0$. Then $(0 :_M P) \subseteq (0 :_M r)$. Since M is a comultiplication semimodule, we have

$$M = (PM :_M P) \subseteq ((0 :_M Ann_R(PM)) :_M P) = ((0 :_M P) :_M Ann_R(PM))$$
$$\subseteq ((0 :_M r) :_M Ann_R(PM)) = (PM :_M r).$$

It follows that $rM \subseteq PM$. This implies that $r \in P$ by [4, Proposition 2.3]. Therefore, $P = Ann_R((0 :_M P))$ because the reverse inclusion is clear. □

Proposition 2.9. *Let S be a second subsemimodule of a comultiplication R-semimodule M and let $N_1, \dots, N_t$ be subsemimodules of M. Then the following statements are equivalent:*

(a) *$S \subseteq N_j$ for some j with $1 \leqslant j \leqslant t$;*

(b) *$S \subseteq \sum_{i=1}^{t} N_i$.*

Proof. $(a) \Rightarrow (b)$. This is clear.

$(b) \Rightarrow (a)$. As M is a comultiplication R-semimodule, we have $N_j = (0 :_M Ann_R(N_j))$ for all j with $1 \leqslant j \leqslant n$. Thus $S \subseteq \sum_{i=1}^{t} N_i$ implies that

$$S \subseteq \sum_{i=1}^{t} (0 :_M Ann_R(N_i)) = (0 :_M \cap_{i \in I} Ann_R(N_j))$$

by [10, Lemma 4.4.]. Hence, $\cap_{i \in I} Ann_R(N_j) \subseteq Ann_R(S)$. Now by using Proposition 2.6 (a) and [12, Lemma 2.4], $Ann_R(N_j) \subseteq Ann_R(S)$ for some j with $1 \leqslant j \leqslant n$. This implies that $(0 :_M Ann_R(S)) = S \subseteq N_j = (0 :_M Ann_R(N_j))$. □

Let M, N be R-semimodules, and f be a map from M to N. f is said to be a *semimodule homomorphism* (see [6]) if

(1) $f(x + y) = f(x) + f(y)$ for all $x, y \in M$;

(2) $f(rx) = rf(x)$ for all $r \in R$, $x \in M$.

$Ker(f) := \{a \in M | f(a) = 0\}$ is called the *kernel of* f. Also, $f(M) := \{f(a) | a \in M\}$. It is easy to see that $Ker(f)$ is a subsemimodule of M and $f(M)$ is a subsemimodule of N. A semimodule homomorphism $f : M \to N$ is said to be *k-regular* (*kernel-regular*) if $f(x_1) = f(x_2)$, then $x_1 + k_1 = x_2 + k_2$ for some $k_1, k_2 \in Ker(f)$ [8].

Proposition 2.10. *Let $f : M \to M'$ be a homomorphism of R-semimodules with $Ker(f) = 0$, where M is a subtractive semimodule.*

(*a*) *If S is a second subsemimodule of M and f is k-regular, then $f(S)$ is a second subsemimodule of M'.*

(*b*) *If S' is a second subsemimodule of $f(M)$, then $f^{-1}(S')$ is a second subsemimodule of M.*

Proof. (*a*). Let S be a second subsemimodule of M. If $f(S) = 0$, then $f^{-1}(f(S)) = S + Ker(f) = S = 0$ by [8, Proposition 3.2 (ii)] because f is k-regular and M is subtractive. This contradiction shows that $f(S) \neq 0$. Now assume that $af(S) \neq f(S)$ and $af(S) \neq 0$ for some $a \in R$. Then we have $aS \neq S$ and $aS \neq 0$. These are contradictions.

(*b*). Let S' be a second subsemimodule of $f(M)$. If $f^{-1}(S') = 0$, then $S' = S' \cap f(M) = f(f^{-1}(S')) = 0$. This contradiction shows that $f^{-1}(S') \neq 0$. Now, let $af^{-1}(S') \neq f^{-1}(S')$ and $af^{-1}(S') \neq 0$ for some $a \in R$. Then $aS' \neq S'$ and $aS' \neq 0$. These are contradictions. □

Corollary 2.11. *If N and K are subsemimodules of an R-semimodule M with N is subtractive, $K \subseteq N \subseteq M$, and K is a second subsemimodule of N, then K is a a second subsemimodule of M.*

Proposition 2.12. *Let S be a subsemimodule of an R-semimodule M such that $Ann_R(S)$ is a maximal ideal of R. Then S is a second subsemimodule.*

Proof. As $Ann_R(S) \neq 0$, we have $S \neq 0$. Let $r \in R$. If $rS \neq 0$, then $Ann_R(S) \subseteq Ann_R(S) + Rr \subseteq R$ implies that $Ann_R(S) + Rr = R$. Thus $Ann_R(S)S + rS = S$. Hence $rS = S$, as needed. □

Proposition 2.13. *Let $\mathfrak{p} \in Spec(R)$. Then the following hold:*

(*a*) *The sum of $\mathfrak{p}$-second semimodules is a $\mathfrak{p}$-second semimodule.*

(*b*) *Every product of $\mathfrak{p}$-second semimodule is a $\mathfrak{p}$-second semimodule.*

(*c*) *Every non-zero quotient of a $\mathfrak{p}$-second semimodule is likewise $\mathfrak{p}$-second.*

Proposition 2.14. *Let M be an R-semimodule. If every non-zero subsemimodule of M is second, then for each subsemimodule K of M and each ideal I of R, we have $(K :_M I) = (K :_M I^2)$. Also for any two ideals A, B of R, $(K :_M A)$ and $(K :_M B)$ are comparable.*

Definition 2.15. We say that a subsemimodule N of an R-semimodule M is *coidempotent* if $N = (0 :_M Ann^2_R(N))$. Also, an R-semimodule M is said to be *fully coidempotent* if every subsemimodule of M is coidempotent.

Proposition 2.16. *Let M be a fully coidempotent R-semimodule.*

- (a) *M is a comultiplication semimodule.*
- (b) *Every subsemimodule and every homomorphic image of M is fully coidempotent.*
- (c) *M is Hopfian.*

Proof. (a). This is clear.

(b). It is easy to see that every subsemimodule of M is fully coidempotent. Now let N be a submodule of M and K/N be a subsemimodule of M/N. By part (a), M is a comultiplication R-semimodule. Hence $K = (0 :_M Ann^2_R(K))$ implies that $K = (0 :_M Ann^3(K))$. Thus

$$(0 :_{M/N} Ann^2(K/N)) = (0 :_M Ann_R(N)Ann^2_R(K/N))/N$$
$$\subseteq (0 :_M Ann^3_R(K))/N = K/N.$$

Therefore, $K/N = (0 :_{M/N} Ann^2_R(K/N))$.

(c). Let $f : M \to M$ be an epimorphism. Then by assumption and part (a), $Ker(f) = (0 :_M I) = (0 :_M I^2)$, where $I = Ann_R(ker(f))$. If $y \in Ker(f)$, then $y \in (0 :_{f(M)} I)$ because f is epic. Thus $y = f(x)$ for some $x \in M$ and $f(x)I = 0$. Hence $xI^2 = 0$. It follows that $xI = 0$. Therefore, $y = 0$, as required. □

Theorem 2.17. *Let M be a fully coidempotent R-semimodule. Then every second subsemimodule of M is a minimal subsemimodule of M.*

Proof. Let S be a second subsemimodule of M and K be a subsemimodule of S. If $Ann_R(K) \subseteq Ann_R(S)$, then $S \subseteq K$ because M is a comultiplication R-semimodule by Proposition 2.16 (a). If $Ann_R(K) \not\subseteq Ann_R(S)$, then there exists $r \in Ann_R(K) - Ann_R(S)$. Since S is second $rS = S$. By Proposition 2.16 (b), S is fully coidempotent. Hence by Proposition 2.16 (c), S is Hopfian. It follows that the epimorphism $r : S \to S$ is an isomorphism. Hence $rK = 0$ implies that $K = 0$, as required. □

Lemma 2.18. *Let $R = R_1 \times R_2$, where R_j is a commutative semiring for all $j \in \{1,2\}$ and J_1 is an ideal of R_1. If J_1 is a prime ideal of R_1, then $J_1 \times R_2$ is a prime ideal of R.*

Proof. Let J_1 be a prime ideal of R_1 and $(a,b)(c,d) \in J_1 \times R_2$. Then $ac \in J_1$. Since J_1 is a prime ideal of R_1, $a \in J_1$ or $c \in J_1$. Thus $(a,b) \in J_1 \times R_2$ or $(c,d) \in J_1 \times R_2$. Hence, $J_1 \times R_2$ is a prime ideal of R. The proof of the converse is trivial. □

Theorem 2.19. *Let $R = R_1 \times R_2$, where R_j is a commutative semiring for all $j \in \{1,2\}$ and J_1 is an ideal of R_1 and J_2 is an ideal of R_2 such that $J = J_1 \times J_2$ is an ideal of R. Then the following statements are equivalent:*

(a) *J is a prime ideal of R;*

(b) *$J = J_1 \times R_2$ for some prime ideal J_1 of R_1 or $J = R_1 \times J_2$ for some prime ideal J_2 of R_2.*

Proof. $(a) \Rightarrow (b)$. Let $J = J_1 \times J_2$ be a prime ideal of R, where J_1 is an ideal of R_1 and J_2 is an ideal of R_2. Then $(0,1)(1,0) \in J_1 \times J_2 = J$ implies that $(1,0) \in J_1 \times J_2$ or $(0,1) \in J_1 \times J_2$. Therefore, $J_1 = R$ or $J_2 = R$. Thus $J = R_1 \times J_2$ or $J = J_1 \times R_2$. Assume that $J = J_1 \times R_2$ for some proper ideal J_1 of R_1. Now, we show that J_1 is a prime ideal. Assume contrary that J_1 is not a prime ideal of R_1. Then there exist elements $x, y \in R_1$ such that $xy \in J_1$ but neither $x \in J_1$ nor $y \in J_1$. Thus, $(x, 1_{R_2})(y, 1_{R_2}) \in J_1 \times R_2$ implies that $(x, 1_{R_2}) \in J_1 \times R_2$ or $(y, 1_{R_2}) \in J_1 \times R_2$. Consequently, $x \in J_1$ or $y \in J_1$, which gives a contradiction. Hence, J_1 is a prime ideal of R_1.

$(a) \Rightarrow (b)$. This follows from Lemm 2.18. □

Lemma 2.20. *Let R_i be a commutative semiring with identity 1_{R_i} and M_i be a faithful R_i-semimodule, for $i = 1,2$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$. Suppose that S_i is a subsemimodule of M_i for $i = 1,2$ such that $S = S_1 \times S_2$ is a subsemimodule of M. Then the followings are equivalent:*

(a) *S is a second subsemimodule of M;*

(b) *S_1 is a second subsemimodule of M_1 and $S_2 = 0$ or $S_1 = 0$ and S_2 is a second subsemimodule of M_2.*

Proof. $(a) \Rightarrow (b)$. Clearly, M is a faithful R-module. By Proposition 2.6, $Ann_R(S)$ is a prime ideal of R. Thus we have either $Ann_{R_1}(S_1) = R_1$ or $Ann_{R_2}(S_2) = R_2$ by Theorem 2.19. So we can assume that $Ann_{R_1}(S_1) = R_1$. Then $S_1 = 0$. Now we prove that S_2 is a second subsemimodule of M_2.

To see this, let $a_2 \in R_2$. Then by assumption $(0, a_2)S = 0$ or $(0, a_2)S = S$. Thus we have $a_2S_2 = 0$ or $a_2S_2 = S_2$, as needed.

$(b) \Rightarrow (a)$. Assume that $S_2 = 0$ and S_1 is a second subsemimodule of M_1. We show that S is a second subsemimodule of M. So let $(a_1, a_2) \in R_1 \times R_2$. Then $a_1S_1 = 0$ or $a_1S_1 = S_1$. Thus $(a_1, a_2)(S_1 \times 0) = 0$ or $(a_1, a_2)(S_1 \times 0) = S_1 \times 0$. Hence, S is a second subsemimodule of M. □

Theorem 2.21. *Let R_i be a commutative semiring with identity 1_{R_i} and M_i be a faithful R_i-semimodule, for $i = 1, 2, \ldots, n$ where $n \geq 2$. Let $R = R_1 \times R_2 \times \cdots \times R_n$, $M = M_1 \times M_2 \times \cdots \times M_n$, and $S = S_1 \times S_2 \times \cdots \times S_n$, where S_i is a subsemimodule of M_i, $1 \leqslant i \leqslant n$. Then the followings are equivalent:*

- (a) *S is a second subsemimodule of M;*
- (b) *S_j is a second subsemimodule of M_j for some $j \in \{1, 2, \ldots, n\}$ and $S_i = 0$ for each $i \neq j$.*

Proof. We use induction on n. By Lemma 2.20, the claim is true if $n = 2$. So, suppose that the claim is true for each $k \leqslant n-1$ and let $k = n$. Put $Q = S_1 \times S_2 \times \cdots \times S_{n-1}$, $\acute{R} = R_1 \times R_2 \times \cdots \times R_{n-1}$, and $\acute{M} = M_1 \times M_2 \times \cdots \times M_{n-1}$, by Lemma 2.20, $S = Q \times S_n$ is a second subsemimodule of $M = \acute{M} \times M_n$ if and only if Q is a second subsemimodule of $\acute{M}$ and $S_n = 0$ or $Q = 0$ and S_n is a second subsemimodule of M_n. Now the rest follows from induction hypothesis. □

An R-semimodule M is said to be *Noetherian* if M satisfies the ACC on its R-subsemimodules. Also, M is said to be *Artinian* if M satisfies the DCC on its R-subsemimodules [1].

An R-semimodule M is said to be *simple* if it has no proper subsemimodules. Also, M is said to be *semisimple* if it is a direct sum of its simple subsemimodules [8].

Proposition 2.22. *Let M be a finitely generated second R-semimodule. Then $Ann_R(M)$ is a maximal ideal of R.*

Proof. Since M is second, $M \neq 0$. There exists a maximal submodule U of M because M is finitely generated by [8, Proposition 2.1]. Clearly, $Ann_R(M) \subseteq Ann_R(M/U)$. If $Ann_R(M/U) \nsubseteq Ann_R(M)$, then there is $r \in R$ such that $rM \subseteq U$ and $rM \neq 0$. This implies that $rM = M$ since M is second. Therefore, $U = M$ which is a contradiction since U is maximal. Thus $Ann_R(M) = Ann_R(M/U)$. Now, as M/U is simple, $Ann_R(M) = Ann_R(M/U)$ is a maximal ideal of R. □

Definition 2.23. Let N be a subsemimodule of an R-semimodule M. We define the *socle* of N as the sum of all second subsemimodules of M contained in N and it is denoted by $sec(N)$. In case N does not contain any second subsemimodule, the socle of N is defined to be (0). Also, we say that $N \neq 0$ is a *socle subsemimodule of* M if $sec(N) = N$.

Theorem 2.24. *Let M be a Noetherian R-semimodule. Then there is a second subsemimodule of M which contains each socle subsemimodule of M.*

Proof. If M does not contain any second subsemimodules, then the results is true vacuously. So we assume that M contains a second subsemimodule. Let $\sum$ be the set of all subsemimodules of M which can be expressed as a sum of a finite number of second subsemimodules. Since M is Noetherian, $\sum$ has a maximal member, S say. Hence, there exist second subsemimodules $S_1, S_2, ..., S_n$ of M such that $S = S_1 + S_2 + ... + S_n$. Let L be any second subsemimodule of M. Then

$$S = S_1 + S_2 + ... + S_n \subseteq L + S_1 + S_2 + ... + S_n \in \sum .$$

By the maximality of S, we have $S = L + S_1 + S_2 + ... + S_n$. Hence, $L \subseteq S$. Thus, S contains each socle subsemimodule of M. □

Definition 2.25. Let M be an R-semimodule. We say that a second subsemimodule N of M is a *maximal second subsemimodule* of a subsemimodule K of M, if $N \subseteq K$ and there does not exist a second subsemimodule L of M such that $N \subset L \subset K$.

Theorem 2.26. *Let M be an R-semimodule. If M satisfies the descending chain condition on socle subsemimodules, then every non-zero subsemimodule of M has only a finite number of maximal second subsemimodules.*

Proof. Suppose that there exists a non-zero subsemimodule N of M such that N has an infinite number of maximal second subsemimodules and look for a contradiction. Then $sec(N)$ is a socle subsemimodule of M and it has an infinite number of maximal second subsemimodules. Let S be a socle subsemimodule of M chosen minimal such that S has an infinite number of maximal second subsemimodules. Then S is not second. Thus there exists a subsemimodule L of M and an ideal I of R such that $L \subset S$ and $S \nsubseteq (0 :_M I)$. Let V be a maximal second subsemimodule of M contained in S. Then $V \subseteq (0 :_S I)$ or $V \subseteq L$. By the choice of S, both the semimodules

$(0:_S I)$ and L have only finitely many maximal second subsemimodules. Therefore, there is only a finite number of possibilities for the semimodule S, which is a contradiction. □

Corollary 2.27. *Every Artinian R-semimodule contains only a finite number of maximal second subsemimodules.*

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