Quasigroups and Related Systems **34** (2026), 111 − 130

https://doi.org/10.56415/qrs.v34.09

Mate I Loops

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Abstract. A loop $(Q,\cdot)$ is called a Mate I loop if it obeys the identity $(x\cdot xy)z = (yz\cdot x)x$. Interestingly, a Mate I loop is a left central loop (LC-loop) but not necessarily a central loop or an extra loop (and vice versa). In this paper, it is shown that a loop $(Q,\cdot)$ is a Mate I loop if and only if it is a LC-loop and a 2can IV loop. Furthermore, we show that a centrum square loop is a Mate I loop if and only if it is a central loop. A Mate I loop is also shown to be a 2can IV loop, a Triad IV loop and a Triad VII loop. Its relationship with some other loops of perfect type is established. Additionally, we proved that a left Bol loop that is also a left Cheban loopis a Mate I loop, and in a Mate I loop, the left Bol law, the LCC law and the left Cheban law are equivalent. Importantly, a fascinating discovery in this study is the fact that a Mate I loop was found to possess two particular properties which two new classes of power associative left (right) conjugacy closed loops possess (each characterizes weak inverse property, power associative conjugacy closed loop).

1. Introduction

A loop $(Q,\cdot)$ is a set together with a binary operation $\cdot$ such that for any $x\in Q$, the right translation map $R_x : Q\to Q; y\mapsto yx$ and the left translation map $R_x : Q\to Q; y\mapsto xy$ are bijections and there exists $e\in Q$ such that $ex = xe = x$ for any $x\in Q$. Since the translation maps are bijections, then the inverse maps R_x^{-1} and L_x^{-1} exist and are thus defined by $yR_x^{-1} = y/x = xM_y^{-1}$ and $yL_x^{-1} = x\backslash y = xM_y$. For an overview of the theory of loops, readers may check [2, 3, 5, 13, 20, 25].

The group formed by all permutations on a set G is called the permutation group of G and denoted by $SYM(G)$. The group $\mathcal{M}(G,\cdot) = \langle\{R_x, L_x, \ : \ x\in G\}\rangle$ is called the multiplication group of the loop $(G,\cdot)$.

2010 Mathematics Subject Classification: Primary 20N02, 20N05
Keywords: Mate I loop, LC loop, C-loop, loops of generalized Bol-Moufang type, perfect varieties

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping and they form a group $\mathrm{Inn}(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1} \ , L_{(x,y)} = L_x L_y L_{yx}^{-1} \text{ and } T_x = R_x L_x^{-1}$$

respectively generate the left inner mapping group $\mathrm{Inn}_\lambda(G)$, the right inner mapping group $\mathrm{Inn}_\rho(G)$ and the middle inner mapping group $\mathrm{Inn}_\mu(G)$. The triple (A, B, C) of bijections of a loop $(Q, \cdot)$ is called an autotopism if

$$xA \cdot yB = (x \cdot y)C, \ \forall \ x, y \in Q.$$

Such triples form a group $AUT(Q, \cdot)$ called the autotopism group of $(Q, \cdot)$.

The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop Q are defined by

$$\begin{aligned} N_\lambda(Q) &= \{a \in Q : \ a \cdot xy = ax \cdot y \ \ \forall \ x, \ y \in Q\}, \\ N_\mu(Q) &= \{a \in Q : \ xa \cdot y = x \cdot ay \ \ \forall \ x, \ y \in Q\}, \\ N_\rho(Q) &= \{a \in Q : \ xy \cdot a = x \cdot ya \ \ \forall \ x, \ y \in Q\}. \end{aligned}$$

The intersection

$$N(Q) = N_\rho(Q) \cap N_\lambda(Q) \cap N_\mu(Q)$$

is called the nucleus of Q.

The commutant of Q is given by $C(Q) = \{c : \forall x \in Q, cx = xc\}$. A loop Q is said to have the centrum square property (CSP) if $x^2 \in C(Q)$.

The center of a loop $Z(Q)$ is defined by $Z(Q) = N(Q) \cap C(Q)$.

A loop has the square symmetric property (SSP) if it satisfies $x \cdot xy = yx \cdot x$.

A loop is a weak inverse property loop (WIPL) if it satisfies any of the following identities: $x(yx)^\rho = y^\rho \quad or \quad (xy)^\lambda x = y^\lambda$.

A loop Q is said to have the cross inverse property if any two elements $x, y \in Q$ satisfy $xy \cdot x^\rho = y$ or $x^\lambda \cdot yx = y$.

A loop Q satisfies the left inverse property (LIP) if $x^\lambda \cdot xy = y$ and the right inverse property (RIP) if $xy \cdot y^\rho = x$. An inverse property loop is a loop that satisfies both the (LIP) and the (RIP).

A loop is said to be conjugacy closed (CC-loop) if it satisfies the two identities:

$$(xy)/x \ \cdot xz = x(yz) \qquad \text{(LCC)} \qquad zx \ \cdot x\backslash(yx)(zy)x. \qquad \text{(RCC)}$$

A loop $(Q,\cdot)$ is called a left central loop (LC-loop) if it satisfies the following identity for all $x,y,z\in Q$:

$$(x\cdot xy)z=x(x\cdot yz). \tag{1}$$

A loop $(Q,\cdot)$ is called a right central loop (RC-loop) if for all $x,y,z\in Q$ it satisfies the identity

$$y(zx\cdot x)=(yz\cdot x)x. \tag{2}$$

$(Q,\cdot)$ is called a central loop (C-loop) if it satisfies the identity

$$(yx\cdot x)z=y(x\cdot xz). \qquad \text{(C)}$$

$(Q,\cdot)$ is called an extra loop if it satisfies the identity

$$(y\cdot xz)x=yx\cdot zx. \qquad \text{(E)}$$

A loop $(Q,\cdot)$ is a Steiner loop if $x^2=e$, $yx\cdot x=y$ and $xy=yx$.

A loop $(Q,\cdot)$ is called a left Bol (resp. right Bol) loop if for all $x,y,z\in Q$ it satisfies

$$(x\cdot yx)z=x(y\cdot xz) \qquad \text{(LB)} \qquad (yx\cdot z)x=y(xz\cdot x). \qquad \text{(RB)}$$

A loop $(Q,\cdot)$ is called a Moufang loop if for all $x,y,z\in Q$ any of the following identities is satisfied

$$(xz\cdot x)y=x(z\cdot xy) \quad \text{(LM)} \quad xy\cdot zx=x(yz\cdot x) \quad \text{(MM1)}$$
$$(yx\cdot z)x=y(x\cdot zx) \quad \text{(RM)} \quad xy\cdot zx=(x\cdot yz)x. \quad \text{(MM2)}$$

The LC-loops, RC-loops, C-loops, extra loops, left (right) Bol loops and the Moufang loops are among the varieties of loops of first Bol-Moufang type (Bol-Moufang type). These varieties of loops have some common features:

1. the only binary operation is multiplication,

2. it has 3 distinct variables with two appearing once on both sides, the third variable appears 2 times,

3. the variables appear in the same order on both sides of the equal sign.

The classification of loops of Bol-Moufang type was first done by Fenyves [7, 8] and completed by Phillips and Vojtěchovský [23]. The four Moufang identities described above have been shown to be equivalent to the following identities:

$$(yx \cdot zx)x = y((xz \cdot x)x) \quad (Q_5) \quad (xy \cdot zx)x = x((yz \cdot x)x) \quad (Q_6)$$
$$x(xy \cdot xz) = (x(x \cdot yx))z \quad (Q_{10}) \quad x(xy \cdot zx) = (x(x \cdot yz))x \quad (Q_9)$$

in [11]. Notably, these identities were introduced alongside eight other identities, collectively referred to as identities of the second Bol-Moufang type. Some of these identities have been studied in [9, 10, 12, 17].

Cote et.al. [4] introduced loops of a generalized Bol-Moufang type by dropping the third condition in the definition of Bol-Moufang type. By generalized Bol-Moufang type, we mean those identities in which the variables do not appear in the same order on both sides of the equal sign. There are exactly 48 varieties of generalized Bol-Moufang type loops, which includes 14 varieties of Bol-Moufang type, 6 varieties of commutative Bol-Moufang type and 28 new varieties (Table 1.2). Out of these 28 varieties, only the Frute loop [6, 15, 16] and the Cheban varieties [19, 22] have been investigated.

We now recall some existing results in literature which are relevant to the current study.

Theorem 1.1. [8, 24] *Let $(Q, \cdot)$ be a LC-loop (RC-loop). Then:*

(i) *$(Q, \cdot)$ is a left (right) alternative loop,*

(ii) *$(Q, \cdot)$ is a left (right) inverse property loop,*

(iii) *$(Q, \cdot)$ is a left (right) nuclear square loop,*

(iv) *$(Q, \cdot)$ is a left (right) power alternative loop.*

Theorem 1.2. [1, 8, 14, 24]

1. *A loop is a C-loop if and only if*
   (a) *it is an LC-loop and RC-loop,*
   (b) *it is an LC-loop (RC-loop) and has RAP (LAP) or RIP (LIP).*
2. *A loop is a extra loop if and only if*
   (a) *it is a Moufang loop and a C-loop,*
   (b) *it is a Moufang loop and a CC-loop.*

Theorem 1.3. [24] *Let* $(Q, \cdot)$ *be a C-loop. Then* $(Q, \cdot)$ *is a both left and right alternative loop, inverse property loop and nuclear square loop.*

The purpose of this work is to investigate the algebraic properties of the Mate I loop which is one of the 28 new varieties.

Variety	Short	Defining Identity	Its Name
Cheban 1 (left Cheban)	C1	$x((xy)z) = (yx)(xz)$	A2C3
Cheban 2 (Cheban)	C2	$x((xy)z) = (y(zx))x$	A2F4
Lonely I	L1	$(x(xy))z = y((zx)x)$	A4F2
Cheban I Dual (right Cheban)	CD	$(yx)(xz) = (y(zx))x$	C3F4
Lonely II	L2	$(x(xy))z = y((xx)z)$	A4C2
Lonely III	L3	$(y(xx))z = y((zx)x)$	C4F2
Mate I	M1	$(x(xy))z = ((yz)x)x$	A4F5
Mate II	M2	$(y(xx))z = ((yz)x)x$	C4F5
Mate III	M3	$x(x(yz)) = y((zx)x)$	A1F2
Mate IV	M4	$x(x(yz)) = y((xx)z)$	A1C2
Triad I	T1	$(xx)(yz) = y(z(xx))$	A3F1
Triad II	T2	$((xx)y)z = y(z(xx))$	A5F1
Triad III	T3	$((xx)y)z = (yz)(xx)$	A5F3
Triad IV	T4	$((xx)y)z = ((yz)x)x$	A5F5
Triad V	T5	$x(x(yz)) = y(z(xx))$	A1F1
Triad VI	T6	$(xx)(yz) = (yz)(xx)$	A3F3
Triad VII	T7	$((xx)y)z = ((yx)x)z$	A5C5
Triad VIII	T8	$(xx)(yz) = y((zx)x)$	A3F2
Triad IX	T9	$(x(xy))z = y(z(xx))$	A4F1
2can I	2C1	$x(yx) = y(xx)$	B4C4
2can II	2C2	$(xy)x = y(xx)$	B5C4
2can III	2C3	$x(xz) = z(xx)$	C1F1
2can IV	2C4	$x(xz) = (zx)x$	C1F2
2can V	2C5	$(xx)z = x(zx)$	C2E1
2can VI	2C6	$(xx)z = (xz)x$	C2E2
Frute	FR	$(x(xy))z = (y(zx))x$	A4F4
Crazy Loop	CR	$(x(xy))z = (yx)(xz)$	A4C3
Krypton	KL	$((xx)y)z = (x(yz))x$	A5D4

Table 1: Varieties of Perfect [4]

2. Main Result

A loop is a Mate I loop if it satisfies

$$(x \cdot xy)z = (yz \cdot x)x. \tag{3}$$

Lemma 2.1. *Let Q be a Mate I loop, then:*

$(p1)$ $L_x^2 = R_x^2$. $(p3)$ $L_{x^2} = L_x^2$. $(p5)$ $L_{x^\lambda} = L_x^{-1}$. $(p7)$ $N_\lambda = N_\mu$.

$(p2)$ $L_{x^2} = R_x^2$. $(p4)$ $x^\lambda = x^\rho$. $(p6)$ $x^2 \in N_\lambda$.

Proof.

1. Setting $z = 1$ in the Mate I identity.
2. Put $y = 1$ in the Mate I identity.
3. This is from (p1) and (p2).
4. This is achieved by putting $z = y^\rho$ and $x = y^\lambda$ in the Mate I identity.
5. Put $x = y^\lambda$ in the Mate I identity and use (p1).
6. Use (p2) and (p3) respectively on the right and the left hand side of the Mate I identity.
7. This is from the fact that in a LIP loop, $N_\lambda = N_\mu$. □

Lemma 2.2. *Let Q be a loop. Then Q is a Mate I loop iff Q obeys*

$$(yx \cdot x)z = x(x \cdot yz). \tag{4}$$

Proof. Suppose Q isa Mate I loop, then using Lemma 2.1 (p1), we have (4). Note also that (4) implies $x \cdot xy = yx \cdot x$ and it is easy to see that (4) implies (3). □

Lemma 2.3. *Let $(Q, \cdot)$ be a Mate I loop. Then $(Q, \cdot)$ is a $2can$ IV loop, a Triad IV loop and a Triad VII loop.*

Proof. By Lemma 2.1 (p1), Q satisfies $x \cdot xy = yx \cdot x$, which is precisely $2C4$. For $T4$, use Lemma 2.1 (p3) in the Mate I identity. By Lemma 2.2, the Mate I identity is equivalent to $(yx \cdot x)z = x(x \cdot yz)$, using Lemma 2.1 (p3) and (p6), we have the Triad VII identity. □

Theorem 2.4. *A loop $(Q, \cdot)$ is a Mate I loop iff it is a LC-loop and a 2can IV loop.*

Proof. Suppose Q satisfies the Mate I identity, $(x \cdot xy)z = (yz \cdot x)x$, by Lemma 2.3, Q is a $2C4$ loop. Using this in the Mate I identity, we have the LC identity.

Conversely, suppose Q is a LC-loop and $2C4$ loop. Use $x(xz) = (zx)x$ on the R.H.S of the LC-loop identity, $(x \cdot xy)z = x(x \cdot yz)$ to obtain the Mate I identity. □

Proposition 2.5. *Let Q be Mate I loop with the weak inverse property (WIP). Then $x^2 \in Z(Q)$.*

Proof. Let Q be a Mate I loop. The Mate I identity is equivalent to $(L_x^2,\ I,\ R_x^2)$ being an autotopism of Q. Since Q has WIP, then by [18, Lemma 1], $(I,\ L_x^{-2},\ R_x^{-2})$ is also an autotopism of Q. This autotopism is equivalent to $(yz \cdot x)x = y(x \cdot xz)$ and with $z = 1$, we have $yx \cdot x = y \cdot xx$. Now Q has RAP, LAP and SSP, then $y \cdot xx = yx \cdot x = x \cdot xy = xx \cdot y \Rightarrow x^2 \in C(Q)$ and by Lemma 2.1 (2.1), (2.1), $x^2 \in N_\lambda \cap N_\mu$.

Now, $x^2 \in C(Q) \cap N_\lambda \cap N_\mu$,

$$yz \cdot x^2 = x^2 \cdot yz = x^2 y \cdot z = yx^2 \cdot z = y \cdot x^2 z = y \cdot zx^2.$$

Thus, $x^2 \in N_\rho$ and therefore $x^2 \in C(Q) \cap N(Q) = Z(Q)$. □

Theorem 2.6. *A centrum square loop is a C-loop if and only if it is a Mate I loop.*

Proof. Suppose Q is a C-loop with the centrum square property. By Theorem 1.3, a C-loop has the left and the right alternative property, therefore,

$$x \cdot xy = xx \cdot y = y \cdot xx = yx \cdot x.$$

Thus, Q has the square symmetric property. Starting with the C identity, $(yx \cdot x)z = y(x \cdot xz) \overset{LAP}{\Leftrightarrow} (yx \cdot x)z = y(xx \cdot z) \overset{x^2 \in C(L)}{\Leftrightarrow} (yx \cdot x)z = y(z \cdot xx) \overset{x^2 \in N_\rho}{\Leftrightarrow}$ $(yx \cdot x)z = (yz \cdot xx) \overset{R_x^2 = L_x^2}{\Leftrightarrow} (x \cdot xy)z = (yz \cdot xx) \overset{RAP}{\Leftrightarrow} (x \cdot xy)z = (yz \cdot xx)$. Thus, Q is a Mate I loop.

Conversely, suppose Q is Mate I loop with the centrum square property and by Lemma 2.1 (p2), $xx \cdot y = yx \cdot x$, we therefore have $yx \cdot x = xx \cdot y = y \cdot xx$.

Thus, Q has RAP, this together with the centrum square property and Lemma 2.1 (p6), (p7) imply $x^2 \in N_\rho(Q)$.

With this information, the converse follows by reversing the proof. □

Corollary 2.7.

1. *A Mate I loop is a C-loop* $\Leftrightarrow$ *it is centrum square* $\Leftrightarrow$ $x^2yx^{-1} = yx$.
2. *A C-loop is a Mate I loop* $\Leftrightarrow$ *it is centrum square. Hence, commutative C-loops and Boolean C-loops are centrum square Mate I loops.*
3. *A loop is a Mate I loop and a* RAP (RIP) *loop if and only if it is a C-loop and centrum square if and only if* $x^2 \in Z(Q)$.

Remark 2.8. In [24], the authors established some results on the structures for commutative C-loops. It would be interesting to investigate such for centrum square Mate I loops since commutative C-loops and Boolean C-loops fall into its class.

Corollary 2.9. *A Steiner loop is a Mate I loop.*

Proof. Since a Steiner loop is a C-loop follows from (Lemma 2.1, [24]), then it is an LC-loop. In a Steiner loop, $R_x^2 = I = L_x^2$, so it is a 2can IV loop. Hence, a Mate I loop by Theorem 2.4. □

Theorem 2.10. *Let* Q *be a Mate I loop. Then the following are equivalent:*

1. Q *is left Bol.*
2. Q *is left Cheban.*
3. Q *is LCC.*

Proof. (1) $\Rightarrow$ (2): Suppose Q is a Mate I loop, then by Lemma 2.2, Q equivalently obeys $(yx \cdot x)z = x(x \cdot yz)$ or

$$x\backslash((yx \cdot x)z) = x \cdot yz. \tag{5}$$

The left Bol identity is also equivalent to

$$(x \cdot yx) \cdot x\backslash z = x \cdot yz. \tag{6}$$

From (5) and (6), we have

$$x\backslash((yx \cdot x)z) = (x \cdot yx) \cdot x\backslash z \Rightarrow x\backslash(yx \cdot z) = xy \cdot x\backslash z \Rightarrow yx \cdot xz = x(xy \cdot z).$$

Thus, Q is a left Cheban.

(2) $\Rightarrow$ (1): Suppose Q is a left Cheban,

$$yx \cdot xz = x(xy \cdot z) \Rightarrow x\backslash(yx \cdot z) = xy \cdot x\backslash z \Rightarrow x\backslash((yx \cdot x)z) = (x \cdot yx) \cdot x\backslash z.$$

Since Q is also a Mate I loop, $x \cdot yz = x\backslash((yx \cdot x)z) = (x \cdot yx) \cdot x\backslash z$ or $(x \cdot yx)z = x(y \cdot xz)$.

(1) $\Rightarrow$ (3): Let Q be a Mate I loop and suppose Q is also a left Bol loop. The Mate I identity is equivalent to $xy \cdot z = ((x\backslash y)z \cdot x)x$ and the left Bol identity is equivalent to $xy \cdot z = x(y/x \cdot xz)$, then

$$\begin{aligned} x(y/x \cdot xz) = ((x\backslash y)z \cdot x)x \Rightarrow x(y/x \cdot xz) &= x(x((x\backslash y)z)) \\ &\Rightarrow (xy)/x \cdot xz = x \cdot yz. \end{aligned}$$

Thus, Q is a LCC loop.

(3) $\Rightarrow$ (1): Suppose Q is a LCC loop,

$$\begin{aligned} (xy)/x \cdot xz = x \cdot yz \Rightarrow x(y/x \cdot xz) &= x(x((x\backslash y)z)) \\ &\Rightarrow x(y/x \cdot xz) = ((x\backslash y)z \cdot x)x. \end{aligned}$$

In a Mate I loop, the R.H.S of the last equation is equal to $(x \cdot x(x\backslash y))z$. Therefore, we have

$$(x \cdot x(x\backslash y))z = x(y/x \cdot xz) \Rightarrow xy \cdot z = x(y/x \cdot xz) \Rightarrow (x \cdot yx)z = x(y \cdot xz).$$

(2) $\Leftrightarrow$ (3): Let Q be a Mate I loop, then by Lemma 2.1 (2.1), $L_x^2 = R_x^2$, and by (Theorem 3.1, [22]), a loop is a left Cheban loop if and only if it is a LCC loop and $L_x^2 = R_x^2$. □

Proposition 2.11. *A left Bol loop that is a left Cheban loop is a Mate I loop.*

Proof. Suppose Q is a left Bol that is also a left Cheban loop, then

$$xy \cdot z = x(y/x \cdot xz) = x \setminus (yx \cdot xz).$$

Therefore,

$$x(y/x \cdot xz) = x \setminus (yx \cdot xz)$$

or

$$x(x \cdot yz) = (yx \cdot x)z \Leftrightarrow (x \cdot xy)z = (yz \cdot x)x.$$

□

Theorem 2.12. *Let Q be a Mate I loop. Then Q is a right Cheban loop if and only if Q is an extra loop.*

Proof. Let Q be a Mate I loop and suppose Q is also a right Cheban loop. Replace z with zx in the Mate I identity to obtain $((x\cdot xy)\cdot zx)/x = (y\cdot zx)x$, since Q is also a right Cheban, we must have

$$\begin{aligned} yx\cdot xz &= ((x\cdot xy)\cdot zx)/x, \\ (yx\cdot xz)x &= (yx\cdot x)\cdot zx, \\ (y\cdot xz)x &= yx\cdot zx. \end{aligned}$$

Thus, Q is an extra loop.

Conversely, suppose Q is Mate I and extra, by Lemma 2.1 (p1), a Mate I loop satisfies $x\cdot xy = yx\cdot x$. Replace y with yx in the extra identity and use $yx\cdot x = x\cdot xy$ to obtain

$$\begin{aligned} (yx\cdot xz)x &= (x\cdot xy)\cdot zx, \\ (yx\cdot x(z/x))x &= (x\cdot xy)z, \end{aligned}$$

since Q is Mate I, we therefore must have

$$\begin{aligned} (yx\cdot x(z/x))x &= (yz\cdot x)x, \\ yx\cdot xz &= (y\cdot zx)x. \end{aligned}$$

Q is therefore a right Cheban. □

Lemma 2.13. *Let Q be a Mate I loop. Then Q satisfies:*

(i) $\underbrace{(xy\cdot x)x = x(yx\cdot x)}_{P_\lambda(x,y)}$ *for all* $x,y\in Q$.

(ii) $\underbrace{x(x\cdot yx) = (x\cdot xy)x}_{P_\rho(x,y)}$ *for all* $x,y\in Q$.

Proof. Let Q be a Mate I loop, then by Lemma 2.1 (p1), Q satisfies $x\cdot xy = yx\cdot x$, replace y with xy to obtain

$$\begin{aligned} x(x\cdot xy) &= (xy\cdot x)x, \\ x(yx\cdot x) &= (xy\cdot x)x. \end{aligned}$$

The other identity can be obtained by mirror argument of the proof of the first. □

Remark 2.14. The LWPC and the RWPC loops were introduced by Phillips (2006, [21]). He proved that a loop satisfies LWPC and RWPC if and only if it is a weak inverse property power associative conjugacy closed (WIP PACC) loop. George et al. (2022, [9]) showed that each of these two loops is power associative. They established that the following equations of identities are true in loops

$$(\mathrm{LWPC}) = (\mathrm{LCC}) + (\mathrm{P}_\lambda) \text{ and } (\mathrm{RWPC}) = (\mathrm{RCC}) + (\mathrm{P}_\rho), \text{ where}$$
$$\underbrace{(xy \cdot x)x = x(yx \cdot x)}_{P_\lambda} \qquad \underbrace{x(x \cdot yx) = (x \cdot xy)x}_{P_\rho}$$

The identities P_λ and P_ρ are respectively the identities (i) and (ii) of Lemma 2.13. George and Jaíyéọlá (2022, [11]) investigated two identities (Q_{12} and Q_7) similar to LWPC and RWPC, and christened LTWC and RTWC respectively.

Similarly, they showed that the following equations of identities are true in loops

$$(\mathrm{LTWC}) = (\mathrm{LCC}) + (\mathrm{P}_\rho) \text{ and } (\mathrm{RTWC}) = (\mathrm{RCC}) + (\mathrm{P}_\lambda).$$

Theorem 2.15. *A Mate I loop is a Moufang loop if and only if it is a Cheban loop.*

Proof. Suppose Q is Mate I loop and a Moufang loop (MM2), then

$$((x\backslash y)z \cdot x)x = xy \cdot z = (x(y(z/x)))x \Rightarrow ((x\backslash y)z \cdot x = x(y(z/x))$$
$$\Rightarrow (y \cdot zx)x = x(xy \cdot z).$$

Thus, Q is a Cheban loop.

Conversely, suppose Q is Mate I loop that is also a Cheban loop, replace y with xy in the Cheban identity to get $x((x \cdot xy)z) = (xy \cdot zx)x$, using the Mate I identity and Lemma 2.13,

$$x\backslash((xy \cdot zx)x) = (yz \cdot x)x \Rightarrow (xy \cdot zx)x = x((yz \cdot x)x)$$
$$\Rightarrow (xy \cdot zx)x = ((x \cdot yz)x)x \Rightarrow (xy \cdot zx) = (x \cdot yz)x.$$

Thus, Q is a Moufang loop. □

Theorem 2.16. *Let* $(Q, \cdot)$ *be a Mate I loop. Then*

1. $(Q, \cdot)$ *has right inverse property (RIP)* $\Leftrightarrow$ $x^2y \cdot x^{-1} = yx$ $\Leftrightarrow$ *it has right alternate property (RAP)* $\Leftrightarrow$ *it is a Triad III loop* $\Leftrightarrow$ *it is centrum square* $\Leftrightarrow$ *it is a* 2*can III loop* $\Leftrightarrow$ *it is a Triad IX loop* $\Leftrightarrow$ *it*

is a Mate II loop ⇔ it is a Triad VI loop ⇔ it is a Lonely I loop ⇔ it is an RC-loop ⇔ it is a Triad II loop ⇔ it is a Triad IX loop ⇔ it is a Mate IV loop ⇔ it is a C-loop ⇔ it is a Lonely II loop ⇔ it is a Triad VIII loop ⇔ it is a Mate III loop ⇔ it is a Triad V loop ⇔ it is a Triad I loop.

2. $(Q,\cdot)$ *is a group ⇔ it is a Frute loop ⇔ it is a Crazy loop.*

3. $(L,\cdot)$ *is a Krypton loop ⇔ it is commutative flexible ⇔ it is commutative RC-loop ⇔ it is a* $2can$ *V loop ⇔ it is* $x\cdot zx = zx\cdot x$.

4. $(Q,\cdot)$ *is a CIPL ⇔ it is commutative.*

Proof. In some of our proofs, we shall use the following notations. By $\mathbf{A} \overset{\text{ass}}{\Leftrightarrow} \mathbf{B}$, we mean that the expression $\mathbf{A}$ will become the expression $\mathbf{B}$ if and only if expression $\mathbf{A}$ satisfies the associative law. Also, by $\mathbf{A} \overset{\text{com}}{\Leftrightarrow} \mathbf{B}$, we mean that the expression $\mathbf{A}$ will become the expression $\mathbf{B}$ if and only if expression $\mathbf{A}$ satisfies the commutative law. By $\mathbf{A} \overset{F_i}{\Leftrightarrow} \mathbf{B}$, we mean that the expression $\mathbf{A}$ will become the expression $\mathbf{B}$ if and only if expression $\mathbf{A}$ satisfies the law F_i . The F_i laws are defined as follows:

F_{36}: $(yx\cdot x)z = y(xx\cdot z)$ (RC identity)

F_{37}: $(yx\cdot x)z = y(x\cdot xz)$ (C identity)

F_{39}: $(y\cdot xx)z = y(x\cdot xz)$ (LC identity)

F_{41}: $xx\cdot yz = (x\cdot xy)z$ (LC identity)

F_{42}: $xx\cdot yz = (xx\cdot y)z$ (left nucleus identity)

F_{43}: $xx\cdot yz = x(x\cdot yz)$ (LAP)

F_{45}: $(x\cdot xy)z = (xx\cdot y)z$ (LAP)

F_{46}: $(x\cdot xy)z = x(x\cdot yz)$ (LC identity)

F_{48}: $(xx\cdot y)z = x(x\cdot yz)$ (LC identity)

F_{51}: $yz\cdot xx = (yz\cdot x)x$ (RAP)

F_{53}: $yz\cdot xx = y(zx\cdot x)$ (RC identity)

F_{55}: $(yz\cdot x)x = (y\cdot zx)x$ (associative identity)

F_{56}: $(yz\cdot x)x = y(zx\cdot x)$ (RC identity)

F_{57}: $(yz\cdot x)x = y(z\cdot xx)$ (RC identity)

1. $(x\cdot xy)z = (yz\cdot x)x$, by replacing z by x^e, we have,

$$(x\cdot xy)x^e = (yx^e\cdot x)x \overset{F_{45}}{\Leftrightarrow} (xx\cdot y)x^e = (yx^e\cdot x)x \Leftrightarrow (x^2y)x^e = (yx^e\cdot x)x.$$

by replacing x^e by x^{-1}. We have,

$$x^2y \cdot x^{-1} = (yx^{-1} \cdot x)x.$$

by replacing x by x^{-1},

$$(x^{-1})^2y \cdot (x^{-1})^{-1} = (y(x^{-1})^{-1} \cdot x^{-1})x^{-1} \Leftrightarrow (x^{-1})^2y \cdot x = (yx \cdot x^{-1})x^{-1}$$

$$\Leftrightarrow (x^{-1})^2y \cdot x \overset{\text{LAP}}{=} yxx^{-1} \cdot x^{-1} \Leftrightarrow (x^{-1})^2y \cdot x = yx^{-1} \Leftrightarrow x^2y \cdot x^{-1} = yx.$$

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ has RIP $\Leftrightarrow x^2y \cdot x^{-1} = yx$.

Let Q be a Mate I loop, by Lemma 2.1 (1)

$$L_x^2 = R_x^2 \Rightarrow yL_x^2 = yR_x^2 \Leftrightarrow x^2y = yx^2 \Leftrightarrow x^2 \in C(Q).$$

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ has the right alternate property $\Leftrightarrow x^2 \in C(Q)$.

$(x \cdot xy)z = (yz \cdot x)x \overset{F_{45}}{\Leftrightarrow} (xx \cdot y)z = (yz \cdot x)x \overset{F_{51}}{\Leftrightarrow} (xx \cdot y)z = yz.xx \Leftrightarrow xx.yz = yz.xx \Leftrightarrow x^2.yz = yz.x^2 \Leftrightarrow x^2 \in C(Q) \Leftrightarrow$ it is centrum square.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Triad III loop $\Leftrightarrow$ it is centrum square.
$(x \cdot xy)z = (yz \cdot x)x \Leftrightarrow (y \cdot xx)z = (yz \cdot x)x \Leftrightarrow (x \cdot xy)z = (y \cdot xx)z \Leftrightarrow$ 2can III.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(L, \cdot)$ is a Mate II loop $\Leftrightarrow$ 2can III.

$x(x \cdot yz) = (yx \cdot x)z \overset{F_{43}}{\Leftrightarrow} xx \cdot yz = (yx \cdot x)z \Leftrightarrow xx \cdot yz = yz \cdot xx \Leftrightarrow (yx \cdot x)z = yz \cdot xx \Leftrightarrow$ it is a Mate II loop.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Triad VI loop $\Leftrightarrow$ it is a Mate II loop.

$(x \cdot xy)z = (yz \cdot x)x \overset{F_{56}}{\Leftrightarrow} (x \cdot xy)z = y(zx \cdot x) \overset{F_{45}}{\Leftrightarrow} (xx \cdot y)z = y(z \cdot xx) \Leftrightarrow$ it is a Triad II loop.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Lonely I loop $\Leftrightarrow$ it is a RC-loop $\Leftrightarrow$ it is a Triad II loop.

$(x \cdot xy)z = (yz \cdot x)x \overset{F_{57}}{\Leftrightarrow} (x \cdot xy)z = y(z \cdot xx) \Leftrightarrow (yz \cdot x)x = y(z \cdot xx) \Leftrightarrow$ it is a RC-loop.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Triad IX loop $\Leftrightarrow$ it is a RC-loop.

$x(x \cdot yz) = (yx \cdot x)z \Leftrightarrow x(x \cdot yz) = y(xx \cdot z) \overset{F_{36}}{\Leftrightarrow} (yx \cdot x)z = y(xx \cdot z) \Leftrightarrow$ RC-loop.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Mate IV loop $\Leftrightarrow$ it is a RC-loop.

$(x \cdot xy)z = (yz \cdot x)x \Leftrightarrow (x \cdot xy)z = y(xx \cdot z) \Leftrightarrow (yz \cdot x)x = y(xx \cdot z) \Leftrightarrow$ it is Triad VIII.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Lonely II loop $\Leftrightarrow$ it is a Triad VIII loop.

$x(x \cdot yz) = (yx \cdot x)z \Leftrightarrow x(x \cdot yz) = y(zx \cdot x) \Leftrightarrow (yx \cdot x)z = y(zx \cdot x) \Leftrightarrow$ it is Triad VIII.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Mate III loop $\Leftrightarrow$ it is a Triad VIII loop.

$x(x \cdot yz) = (yx \cdot x)z \Leftrightarrow x(x \cdot yz) = y(z \cdot xx) \Leftrightarrow (yx \cdot x)z = y(z \cdot xx) \Leftrightarrow$ it is Triad VIII.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Triad V loop $\Leftrightarrow$ it is a Triad VIII loop.

$x(x \cdot yz) = (yx \cdot x)z \overset{F_{43}}{\Leftrightarrow} xx \cdot yz = (yx \cdot x)z \Leftrightarrow xx \cdot yz = y(z \cdot xx) \Leftrightarrow (yx \cdot x)z = y(z \cdot xx) \Leftrightarrow$ it is Triad VIII.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a Triad I loop $\Leftrightarrow$ it is a Triad VIII loop.

2. Now

$(x \cdot xy)z = (yz \cdot x)x \overset{F_{55}}{\Leftrightarrow} (x \cdot xy)z = (y \cdot zx)x \Leftrightarrow$ Frute Loop $\overset{F_{45}}{\Leftrightarrow} (xx \cdot y)z = (y \cdot zx)x \overset{F_{42}}{\Leftrightarrow} xx \cdot yz = (y \cdot zx)x \Leftrightarrow$it is a Crazy Loop.

Hence, if $(Q, \cdot)$ be a Mate I loop. Then $(Q, \cdot)$ is a associative i.e. it is a group $\Leftrightarrow$ it is a Frute loop $\Leftrightarrow$ it is a Crazy loop.

3. $(x \cdot xy)z = (yz \cdot x)x \overset{F_{45}}{\Leftrightarrow} (xx \cdot y)z = (yz \cdot x)x \Leftrightarrow (xx \cdot y)z = (x \cdot yz)x \Leftrightarrow (yz \cdot x)x = (x \cdot yz)x \Leftrightarrow$ commutative flexible.

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(L, \cdot)$ is a Krypton loop $\Leftrightarrow$ it commutative flexible.

4. From the Mate I loop identity $(x \cdot xy)z = (yz \cdot x)x$, by replacing x by y^{-1}, we have,

$(y^{-1} \cdot y^{-1}y)z = (yz \cdot y^{-1})y^{-1} \Leftrightarrow y^{-1}z = (yz \cdot y^{-1})y^{-1} \overset{\text{com}}{\Leftrightarrow} y^{-1}z = (zy \cdot y^{-1})y^{-1}$

$$\Leftrightarrow y^{-1}z = zy^{-1} \Leftrightarrow (y^{-1})^{-1}z = z(y^{-1})^{-1} \Leftrightarrow yz = zy.$$

Hence, if $(Q, \cdot)$ be a Mate I loop, then $(Q, \cdot)$ is a CIPL $\Leftrightarrow$ it is commutative. □

Corollary 2.17. *A loop is Mate I and Frute if and only if it is a group and central square.*

Proposition 2.18. *Let $(Q, \cdot)$ be a finite loop with right and left representation sets $\Pi_\rho(Q)$ and $\Pi_\lambda(Q)$, then*

1. *$(Q, \cdot)$ is a Mate I loop iff $\alpha\beta^2 \in \Pi_\lambda(Q)$ for every $\alpha \in \Pi_\lambda(Q)$ and $\beta \in \Pi_\rho(Q)$.*
2. *$(Q, \cdot)$ is a Mate I loop iff $\alpha\beta^2 \in \Pi_\lambda(Q)$ for every $\alpha, \beta \in \Pi_\lambda(Q)$.*

Proof. 1. Suppose Q is Mate I loop, i.e.,

$$(x \cdot xy)z = (yz \cdot x)x \Rightarrow L_{x \cdot xy} = L_y R_x^2 \Rightarrow \alpha\beta^2 \in \Pi_\lambda$$

for every $\alpha \in \Pi_\lambda(Q)$ and $\beta \in \Pi_\rho(Q)$.

Conversely, let $\alpha\beta^2 \in \Pi_\lambda$ for every $\alpha \in \Pi_\lambda(Q)$ and $\beta \in \Pi_\rho(Q)$, then for $t \in Q$, $\alpha\beta^2 = L_t$, where $\alpha = L_y, \beta = R_x$ and $t = x \cdot xy$.
Now $\alpha\beta^2 = L_t \Rightarrow L_y R_x^2 = L_{x \cdot xy} \Rightarrow z L_y R_x^2 = z L_{x \cdot xy} \Rightarrow (x \cdot xy)z = (yz \cdot x)x$.

2. This can be done in a similar way by using equation 4. □

Theorem 2.19. *Let $(Q, \cdot)$ be a loop. Then:*

1. *$(Q, \cdot)$ is a Mate I loop iff $(L_x^2, I, R_x^2) \in AUT(Q, \cdot)$ for every $x \in Q$. Hence, $(L_x^{2n}, I, R_x^{2n}) \in AUT(Q, \cdot)$ for every $x \in Q$ and $n \in \mathbb{Z}$.*
2. *$(Q, \cdot)$ is a Mate I loop iff $(R_x^2, I, L_x^2) \in AUT(Q, \cdot)$ for every $x \in Q$. Hence, $(R_x^{2n}, I, L_x^{2n}) \in AUT(Q, \cdot)$ for every $x \in Q$ and $n \in \mathbb{Z}$.*
3. *$(Q, \cdot)$ is a Mate I loop iff $R_y R_x^2 = L_x^2 R_y$ for all $x, y \in Q$.*
4. *$(Q, \cdot)$ is a Mate I loop iff $R_y L_x^2 = R_x^2 R_y$ for all $x, y \in Q$.*

   Hence:

5. *$L_x^{2n} = R_y R_x^{2n} R_y^{-1} = R_y^{-1} R_x^{2n} R_y$ for all $x, y \in Q$ and $n \in \mathbb{Z}$.*
6. *$R_x^{2n} = R_y L_x^{2n} R_y^{-1} = R_y^{-1} L_x^{2n} R_y$ for all $x, y \in Q$ and $n \in \mathbb{Z}$.*
7. *$R_y^{2m} L_x^{2n} = L_x^{2n} R_y^{2m}$, $\forall x, y \in Q, n, m \in \mathbb{Z}$.*

8. $R_y^{2m} R_x^{2n} = R_x^{2n} R_y^{2m},\ \forall x, y \in Q, n, m \in \mathbb{Z}$.

9. $((yz * z)\underbrace{x * x)x \cdots x}_{2n-times} = \big[((y\underbrace{x * x)x \cdots x)}_{2n-times}\big]z * z,$ *where* $* \in \{\cdot, /\}$ *for all* $x, y \in Q$ *and* $n \in \mathbb{Z}$.

10. $\underbrace{x \cdots x(x * x}_{2n-times}(zy * y) = y * y\big[\underbrace{(x \cdots x(x * x}_{2n-times} z)$ *where* $* \in \{\cdot, \backslash\}$ *for all* $x, y \in Q$ *and* $n \in \mathbb{Z}$.

11. $x^{2n}y^2 = y^2x^{2n}$ *or* $(x^{2n}, y^2) = e$ *for all* $x, y \in Q$ *and* $n \in \mathbb{Z}$.

12. $x^{2n}y^{2m} = y^{2m}x^{2n}$ *or* $(x^{2n}, y^{2m}) = e$ *for all* $x, y \in Q$ *and* $n, m \in \mathbb{Z}$.

13. $L_x^{2n} = L_{x^{2n}} = R_x^{2n}$ *for all* $x \in Q$ *and* $n \in \mathbb{Z}$.

14. $[L_x^2, R_y] = I \Leftrightarrow [R_x^2, R_y] = I$ *for all* $x, y \in Q$.

Proof.

1. $(Q, \cdot)$ is a Mate I loop $\Leftrightarrow (x \cdot xy)z = (yz \cdot x)x \Leftrightarrow (L_x^2, I, R_x^2) \in AUT(Q, \cdot)$, so, $(L_x^{2n}, I, R_x^{2n}) \in AUT(Q, \cdot)$ for all $x \in Q$ and $n \in \mathbb{Z}$.

2. $(Q, \cdot)$ is a Mate I loop $\Leftrightarrow (yx \cdot x)z = x(x \cdot yz) \Leftrightarrow (R_x^2, I, L_x^2) \in AUT(Q, \cdot)$, so, $(R_x^{2n}, I, L_x^{2n}) \in AUT(Q, \cdot)$ for all $x \in Q$ and $n \in \mathbb{Z}$..

3. $(Q, \cdot)$ is a Mate I loop $\Leftrightarrow (x \cdot xy)z = (yz \cdot x)x \Leftrightarrow yL_x^2R_z = yR_zR_x^2 \Leftrightarrow L_x^2R_z = R_zR_x^2$.

4. $(Q, \cdot)$ is a Mate I loop $\Leftrightarrow (yx \cdot x)z = x(x \cdot yz) \Leftrightarrow yR_x^2R_z = yR_zL_x^2 \Leftrightarrow R_x^2R_z = R_zL_x^2$.

5. From 3. above, Q is a Mate I loop if and only if $L_x^2 = R_zR_x^2R_z^{-1}$ and from 4. Q is a Mate I loop if and only if $L_x^2 = R_z^{-1}R_x^2R_z$. Therefore, $L_x^2 = R_zR_x^2R_z^{-1} = R_z^{-1}R_x^2R_z \Rightarrow L_x^{2n} = R_zR_x^{2n}R_z^{-1} = R_z^{-1}R_x^{2n}R_z$.

6. This is similar to 5.

7. Put 6. in 5., we have $L_x^{2n} = R_y^2L_x^{2n}R_y^{-2} \Rightarrow R_y^2 = L_x^{2n}R_y^2L_x^{-2n} \Rightarrow R_y^{2m} = L_x^{2n}R_x^{2m}L_x^{-2n} \Leftrightarrow R_y^{2m}L_x^{2n} = L_x^{2n}R_x^{2m}$ for all $x, y \in Q, n, m \in \mathbb{Z}$.

8. This is by putting 5. in 6. in a similar way.

9. From 5.,
$R_zR_x^{2n}R_z^{-1} = R_z^{-1}R_x^{2n}R_z \Leftrightarrow R_z^2R_x^{2n} = R_x^{2n}R_z^2 \Leftrightarrow R_x^{-2n}R_z^{-2} = R_z^{-2}R_x^{-2n}$.
So, $((yz * z)\underbrace{x * x)x \cdots x}_{2n-\text{times}} = \big[((y\underbrace{x * x)x \cdots x)}_{2n-\text{times}}\big]z * z$, where $* \in \{\cdot, /\}$.

10. This can be proved in a manner similar to 8., using 6.

11. Set $y = 1$ in 8.

12. Put $z = 1$ in 10.

13. By 5. and 6., we have $L_x^{2n} = R_x^{2n}$ and the rest follows from the fact that $L_{x^n} = L_x^n$ holds in Mate I loops since they are LC loops.

14. In a Mate I loop, $R_x^2 = L_x^2$, then by 3., $R_yR_x^2 = L_x^2R_y \Leftrightarrow R_yL_x^2 = L_x^2R_y \Leftrightarrow [L_x^2, R_y] = I \Leftrightarrow [R_x^2, R_y] = I$. □

Theorem 2.20. *Let $(Q, \cdot)$ be a Mate I loop. For all $x, y, z \in Q$:*

1. $T_x = R_x^{-1}L_x$.
2. $yT_x = xM_y^{-1}L_x$.
3. $(y, x) = (xy)^{-1}x^2 \cdot xM_y^{-1}$.
4. $x^3z = (xz \cdot x)x$.
5. *$(Q, \cdot)$ is flexible if and only if $x^3z = (x \cdot zx)x$.*
6. *$(Q, \cdot)$ is centrum cube if and only if $(Q, \cdot)$ is a 2can II loop.*
7. $x^4z = (x^2zx)x$.
8. *$(Q, \cdot)$ is centrum quartic if and only if $zx^4 = (x^2zx)x$.*
9. $x^2yx^2 = yx^3 \cdot x$, $x^2 = (yx \cdot x)y^{-1}$.

Proof.

1. In the Mate I identity, put $z = x$ and replace y with y/x to obtain
$$x \cdot x(y/x) = yx \Leftrightarrow x(y/x) = x\backslash(yx) \Leftrightarrow yR_x^{-1}L_x = yT_x.$$

2. Again following the same steps as above, we have
$$x \cdot x(y/x) = yx \Leftrightarrow x(y/x) = x\backslash(yx) \Leftrightarrow xM_y^{-1}L_x = yT_x.$$

3. By 2., we have

$$\begin{aligned} yT_x &= xM_y^{-1}L_x \qquad , \\ x\backslash(yx) &= x\cdot xM_y^{-1}, \\ yx &= x\cdot x(xM_y^{-1}), \\ yx &= x^2(xM_y^{-1}), \\ xy(y,x) &= x^2(xM_y^{-1}), \\ (y,x) &= (xy)\backslash(x^2(xM_y^{-1})), \\ (y,x) &= (xy)^{-1}\cdot x^2(xM_y^{-1}) = (xy)^{-1}x^2\cdot xM_y^{-1}. \end{aligned}$$

4. Set $y = x$ in the Mate I identity.
5. Use flexible law in 4.
6. Put $y = x$ and use the 2C4 identity to get

$$(x\cdot xx)z = (xz\cdot x)x = (z\cdot xx)x \Rightarrow x^3z = zx^3.$$

7. Set $y = x^2$ in the Mate I identity.
8. A loop is centrum quartic if and only if $x^4z = zx^4$, the result thus follows from 7.
9. Put $z = x^2$ in the Mate I identity, $(x^2y)x^2 = (yx^2\cdot x)x = yx^3\cdot x$. Put $z = y^\rho$ in the Mate I identity, $(x\cdot xy)y^\rho = x^2$, by Lemma 2.1 p1. and p4., we have $(yx\cdot x)y^{-1} = x^2$. □

□

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Received July 17, 2005

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