

## Isotopic classes of transversals in finite simple groups

*Chandan Singh Kharvar and R.P. Shukla*

**Abstract.** Let  $G$  be a finite group and  $H$  be a subgroup of  $G$ . In this paper, we give a short proof of the main theorem of Kharvar and Shukla [6, Main Theorem]. Further, we prove that if  $H$  is a non-trivial proper subgroup of a finite non-abelian simple group  $G$ , then the number of isotopic classes of NRTs of  $H$  in  $G$  is greater than 1.

### 1. Introduction and tatement of main results

Let  $H$  be a subgroup of group  $G$ . A subset  $S$  of  $G$  is defined as a normalized right transversal (NRT) of  $H$  in  $G$  if it is formed by selecting one and only one element from each right coset of  $H$  in  $G$ , with the condition that  $1 \in S$ , where 1 represents the identity element of the group  $G$ . Let  $S$  be an NRT of  $H$  in  $G$ . Then the binary operation of  $G$  induces a binary operation on  $S$  defined by  $\{x \circ y\} = S \cap Hxy$ ,  $x, y \in S$ . With respect to this binary operation,  $S$  is a right loop with identity 1, that is, a right-quasigroup with both-sided identity (see [11, Proposition 4.3.3]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [7, Theorem 3.4]).

Let  $S$  be an NRT of  $H$  in  $G$ . Let  $\langle S \rangle$  be the subgroup of  $G$  generated by  $S$  and  $H_S$  be the subgroup  $\langle S \rangle \cap H$ . Then  $H_S = \langle \{xy(x \circ y)^{-1} | x, y \in S\} \rangle$  and  $H_S S = \langle S \rangle$  (see [7, Corollary 3.7]). Identifying  $S$  with the set  $H \backslash G$  of all right cosets of  $H$  in  $G$ , we get a transitive permutation representation  $\chi_S : G \rightarrow \text{Sym}(S)$  defined by  $\{\chi_S(g)(x)\} = Hxg \cap S$ ,  $g \in G$ ,  $x \in S$ . The Kernel  $\text{Ker}\chi_S$  of this action is  $\text{Core}_G(H)$ , the core of  $H$  in  $G$ . Our covention for the product in the symmetric group  $\text{Sym}(S)$  is given as  $(rs)(x) = s(r(x))$  for  $r, s \in \text{Sym}(S)$  and  $x \in S$ . Let  $G_S = \chi_S(H_S)$ , the group torsion of the right loop  $S$  (see [7, Definition 3.1, p.75]). Since  $\chi_S$  is injective on  $S$  and if we identify  $S$  with  $\chi_S(S)$ , then  $\chi_S(\langle S \rangle) = G_S S$ .

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Let  $(L, \circ)$  and  $(L', \circ')$  be two groupoids. A triple  $(f, g, h)$ , where  $f, g$  and  $h$  are bijective maps from  $L$  to  $L'$ , is said to be *isotopy* from  $L$  to  $L'$  if  $f(x) \circ' g(y) = h(x \circ y)$ , for all  $x, y$  in  $L$ . Two groupoids  $(L, \circ)$  and  $(L', \circ')$  are said to be *isotopic* if there exists an isotopy from  $L$  to  $L'$ . If  $f = g = h$ , then isotopy  $(f, g, h)$  is an isomorphism from  $L$  to  $L'$ . Let  $(L, \circ)$  and  $(L, \circ')$  be two groupoids. An isotopy  $(f, g, I)$ , where  $I$  is identity map on  $L$ , from  $(L, \circ)$  to  $(L, \circ')$  is said to be a *principal isotope*. A non-empty subset  $T$  of a groupoid  $L$  is called a *subgroupoid* of  $L$  if the binary operation of  $L$  induces the binary operation on  $T$ .

Let  $\mathcal{T}(G, H)$  denote the set of all NRTs of  $H$  in  $G$ . We say that  $S, T \in \mathcal{T}(G, H)$  are isomorphic (resp. isotopic) if their induced right loop structures are isomorphic (resp. isotopic). Let  $Iso(G, H)$  (resp.  $Itp(G, H)$ ) denote the set of isomorphic (resp. isotopic) classes of NRTs of  $H$  in  $G$  respectively.

In [7], it is proved that if  $H$  is a normal subgroup of finite group  $G$ , then  $|Iso(G, H)| = 1$ . It has been proved in [8, Main Theorem] that if  $H$  is a subgroup of finite group  $G$  such that  $|Iso(G, H)| = 1$ , then  $H$  is a normal subgroup of  $G$ . By [3, 4],  $|Iso(G, H)| \neq 2, 4$ . Also, it has been proved in [9, Main Theorem] that if  $G$  is a finite non-abelian nilpotent group non-isomorphic to  $D_8$ , the dihedral group of order 8 and  $H$  is a non-normal subgroup of  $G$ , then  $|Iso(G, H)| > 16$ . It is easy to prove that if  $H$  is a normal subgroup of finite group  $G$ , then  $|Itp(G, H)| = 1$ . A natural question that arises is whether the converse of this problem is true. In [5, Proposition 3.8], it is proved that if  $H$  is a subgroup of finite nilpotent group  $G$  such that  $|Itp(G, H)| = 1$ , then  $H$  is normal in  $G$ . It has been shown in [5, Proposition 3.9] that if  $H$  is a subgroup of finite solvable group  $G$ . If  $(|H|, [G : H]) = 1$  and  $|Itp(G, H)| = 1$ , then  $H$  is normal in  $G$ . In [6, Main Theorem], it is proved that if  $G$  is a finite solvable group and  $H$  is a subgroup of  $G$  such that  $|Itp(G, H)| = 1$ , then  $H$  is normal in  $G$ . In this paper, we give a short proof of the main theorem in [6]

**Theorem 1.1.** [6, Main Theorem] *Let  $G$  be a finite solvable group and  $H$  be a subgroup of  $G$  such that  $|Itp(G, H)| = 1$ . Then  $H$  is normal in  $G$ .*

and prove

**Theorem 1.2.** *Let  $G$  be a finite non-abelian simple group and  $H$  be a non-trivial proper subgroup of  $G$ . Then  $|Itp(G, H)| > 1$ .*

## 2. Preliminaries

In this section, we give some basic definitions and results that are needed to prove results of this paper.

**Definition 2.1.** [1] Let  $(L, \circ)$  be a groupoid. Then the subset

$A = \{a \in L : \text{for all } x, y \in L, (x \circ a) \circ y = x \circ (a \circ y)\}$  of  $L$  is called the *middle-associators* of  $(L, \circ)$ .

**Definition 2.2.** [1] Let  $(L, \circ)$  be a groupoid. Then  $a \in L$  is called a *left non-singular* (resp. *right non-singular*) element of  $L$  if the equation  $a \circ X = b$  (resp.  $X \circ a = b$ ) has unique solution for all  $b \in L$ .

If  $a$  is a left non-singular (resp. right non-singular) element of a groupoid  $(L, \circ)$ , then the map  $L_a : L \rightarrow L$  (resp.  $R_a : L \rightarrow L$ ) defined by  $L_a(x) = a \circ x$  ( resp.  $R_a(x) = x \circ a$ ) is a bijective map.

**Theorem 2.3.** [8, Proposition 1.10] *Let  $G$  be a finite group,  $H$  be a subgroup of  $G$  and  $S \in \mathcal{T}(G, H)$ . Then there exists a surjective homomorphism  $\chi$  from  $\langle S \rangle$  to  $G_S S$ . Further, if  $H$  is corefree, then  $\chi$  is an isomorphism.*

Let  $G$  be a finite group and  $H$  be a subgroup of  $G$ . Let  $Aut_H G$  denote the subgroup of  $Aut(G)$ , the automorphism group of  $G$ , consisting of those automorphisms of  $G$  which take  $H$  onto  $H$ .

**Theorem 2.4.** [8, Proposition 2.7] *Let  $H$  be a corefree subgroup of a finite group  $G$ . Let  $S \in \mathcal{T}(G, H)$  such that  $\langle S \rangle = G$ . Then  $Aut_H G$  acts transitively on the set  $\{T \in \mathcal{T}(G, H) | T \cong S\}$ .*

**Theorem 2.5.** [8, Lemma 3.4] *Let  $G$  be finite non-abelian simple group. Then  $|G|^2 > |Aut(G)|$ , where  $Aut(G)$  is the group of automorphisms of  $G$ .*

**Theorem 2.6.** [1, Theorem 1B] *Let  $(L, \circ)$  be a groupoid. Then the middle-associators of  $(L, \circ)$  is a subgroupoid of  $(L, \circ)$  and is a semigroup. If  $(L, \circ)$  has identity 1, and  $(L, \circ')$  is a groupoid with identity isotopic to  $(L, \circ)$ , then the middle-associators of  $(L, \circ)$  is isomorphic to the middle-associators of  $(L, \circ')$ .*

**Remark 2.7.** Let  $G$  be a finite group,  $H$  be a subgroup of  $G$  and  $S_1, S_2 \in \mathcal{T}(G, H)$ . Let  $A_1, A_2$  be middle-associators of  $S_1, S_2$  respectively. If  $S_1$  is isotopic to  $S_2$ , then by Theorem 2.6,  $A_1$  is isomorphic to  $A_2$ .

**Theorem 2.8.** [6, Proposition 2.3] *Let  $G$  be a finite group,  $H$  be a corefree subgroup of  $G$  and  $S \in \mathcal{T}(G, H)$ . Let  $A$  be the middle-associators of  $(S, \circ)$ , where  $\circ$  is the binary operation on  $S$  induced by the binary operation of  $G$ . Then  $a \in A$  if and only if  $ax \in S$  for all  $x \in S$ , that is, if and only if  $a \circ x = ax$  for all  $x \in S$ .*

**Remark 2.9.** Let  $G$  be a finite group,  $H$  be a corefree subgroup of  $G$  and  $S \in \mathcal{T}(G, H)$ . Let  $A$  be the middle-associators of  $(S, \circ)$ . Then by Theorem 2.8, it follows that the middle-associators  $A$  of  $S$  is a subgroup of  $G$ .

Let  $(L_1, \circ_1)$  and  $(L_2, \circ_2)$  be isotopic groupoids. Then there exists a groupoid  $(L_1, \circ_3)$  isomorphic to  $(L_2, \circ_2)$  and isotopic to  $(L_1, \circ_1)$ . In particular  $(L_1, \circ_3)$  may be chosen to be a principal isotope of  $(L_1, \circ_1)$  [1, Lemma 1A]. Let  $(L_1, \circ_1)$  be a groupoid. Then a necessary and sufficient condition that there exists a groupoid  $(L_2, \circ_2)$ , with an identity, isotopic to  $(L_1, \circ_1)$  is that  $(L_1, \circ_1)$  possess at least one left non-singular element  $a$  and at least one right non-singular element  $b$ . Every principal isotope  $(L_1, \circ_3)$  of the type is given by

$$x \circ_3 y = R_b^{-1}(x) \circ_1 L_a^{-1}(y)$$

for some such fixed pair  $a, b$  of  $(L_1, \circ_1)$  [1, Theorem 1A]. The proof of the following proposition immediately from argument given in [5, Second paragraph of Section 3, p.413].

**Proposition 2.10.** *Let  $G$  be a finite group,  $H$  be a subgroup of  $G$  and  $S_1, S_2 \in \mathcal{T}(G, H)$ . If  $S_1$  and  $S_2$  are isotopic, then  $G_{S_1}S_1 \cong G_{S_2}S_2$ .*

### 3. Proof of the Theorems 1.1 and 1.2

In order to prove our theorems, we need the following propositions and lemma.

**Proposition 3.1.** *Let  $G$  be a finite group and  $H$  be a corefree subgroup of  $G$  such that  $|Itp(G, H)| = 1$ . Let  $S \in \mathcal{T}(G, H)$  and  $A$  be a middle-associators of  $S$ . Then  $A = \{1\}$ .*

*Proof.* We prove this by the method of contradiction. If possible assume that  $A \neq \{1\}$ . Then by Remark 2.9,  $A$  is a subgroup of  $G$ . Let  $p > 2$  be a prime and  $p$  divide  $|A|$ . Then there exists a subgroup  $K$  of  $A$  of order  $p$ . Let  $k(\neq 1) \in K$ ,  $S_1 = (S \setminus \{k\}) \cup \{hk\}$  for some  $h(\neq 1) \in H$  and

$A_1$  be a middle-associators of  $S_1$ . Since  $|K| = p$  and  $k \notin S_1$ , by Remark 2.9,  $k^2, k^3, \dots, k^{p-1}$  are not in  $A_1$ . Since  $k \neq k^{-1}$  and  $(hk)k^{-1} = h \notin S_1$ , by Theorem 2.8,  $hk \notin A_1$ . Since  $|Itp(G, H)| = 1$  and  $p$  divides  $|A|$ ,  $p$  divides  $|A_1|$ . Therefore there exists a subgroup  $K_1$  of  $A_1$  of order  $p$ . Let  $k_1 (\neq 1) \in K_1$ ,  $S_2 = (S_1 \setminus \{k_1\}) \cup \{h_1 k_1\}$  for some  $h_1 (\neq 1) \in H$  and  $A_2$  be a middle-associators of  $S_2$ . By constructions of  $S_2$  and by Theorem 2.8, elements  $hk$  and  $k^2, k^3, \dots, k^{p-1}$  are not in  $A_2$ . Since  $|K_1| = p$  and  $k_1 \notin S_2$ , by Remark 2.9,  $k_1^2, k_1^3, \dots, k_1^{p-1}$  are not in  $A_2$ . Since  $k_1 \neq k_1^{-1}$ ,  $k_1^{-1} \in S_2$ . Since  $(h_1 k_1)k_1^{-1} = h_1$  are not in  $S_2$ , by Theorem 2.8,  $h_1 k_1$  is not in  $A_2$ . In each step we get an NRT of  $H$  in  $G$  with middle associators such that  $p - 1$  elements of NRT are not member of middle associators. Since index of  $H$  in  $G$  is finite, by continuing this process we get an NRT  $S' \in \mathcal{T}(G, H)$  having middle-associators  $A'$  and  $|A'| < p$ . Therefore by Remark 2.7,  $|Itp(G, H)| > 1$ , a contradiction. This implies that only a prime divides the order of  $A$ , which is 2.

Assume that 2 divides  $|A|$ . Then there exists a subgroup  $L$  of  $A$  of order 2. Let  $S = \{1, x_1, x_2, \dots, x_{n-1}\}$ . Without loss of generality we may assume that  $L = \{1, x_1\}$ . Let  $h (\neq 1) \in H$  be fixed and  $T_1 = (S \setminus \{x_1\}) \cup \{hx_1\}$  and  $B_1$  be a middle-associators of  $T_1$ . Since  $x_1 \in A$ , by Theorem 2.8,  $x_1 x_2 \in S$ . By construction of  $T_1$ ,  $x_1 x_2 \in T_1$ . If  $hx_1 \in B_1$ , then by Theorem 2.8,  $hx_1 x_2 \in T_1$ , a contradiction. Hence  $hx_1 \notin B_1$ . Since  $|Itp(G, H)| = 1$  and 2 divides  $|A|$ , 2 also divides  $|B_1|$ . Therefore there exists a subgroup  $L_1$  of  $B_1$  of order 2. Without loss of generality we may assume that  $L_1 = \{1, x_2\}$ . Let  $h (\neq 1) \in H$  be fixed and  $T_2 = (T_1 \setminus \{x_2\}) \cup \{hx_2\}$  and  $B_2$  be a middle-associators of  $T_2$ . Since  $x_1 \in A$  and  $x_2 \in B_1$ , by Theorem 2.8,  $x_1 x_3 \in S$  and  $x_2 x_3 \in T_1$ . By construction of  $T_2$ ,  $x_1 x_3, x_2 x_3 \in T_2$ . If  $hx_1, hx_2 \in B_2$ , then by Theorem 2.8,  $hx_1 x_3, hx_2 x_3 \in T_2$ , a contradiction. Hence  $hx_1, hx_2 \notin B_2$ . Continuing this process we get  $x_1^2 = x_2^2 = x_3^2 = \dots = x_{n-1}^2 = 1$ . Hence by [5, Proposition 2.1],  $S$  is a both sided transversal. Let  $K$  be an NRT of  $H$  in  $G$  and  $K_1$  be a middle-associators of  $K$ . Since  $|Itp(G, H)| = 1$ ,  $|K_1| > 1$ . Hence by previous argument  $K$  is also both sided transversals. Therefore by [5, Theorem 2.6],  $H$  is normal subgroup of  $G$ , a contradiction. This complete the proof.  $\square$

**Proposition 3.2.** *Let  $G$  be a finite group and  $H$  be a corefree subgroup of  $G$  such that  $|Itp(G, H)| = 1$ . Then  $\langle S \rangle = G$ , for all  $S \in \mathcal{T}(G, H)$ .*

*Proof.* Since  $H$  is a corefree subgroup of  $G$ , by [2], there exists  $S \in \mathcal{T}(G, H)$  such that  $\langle S \rangle = G$ . Then by Theorem 2.3,  $G_S S \cong \langle S \rangle = G$ . Let  $T \in$

$\mathcal{T}(G, H)$ . Since  $|Itp(G, H)| = 1$ , then by Theorem 2.3 and Proposition 2.10,  $\langle T \rangle \cong G_T T \cong G_S S \cong \langle S \rangle = G$ . This complete the proof.  $\square$

**Proposition 3.3.** *Let  $G$  be a finite group,  $H$  be a corefree subgroup of  $G$  such that  $|Itp(G, H)| = 1$  and  $[G : H] = n$ . Then  $|Iso(G, H)| \leq n(n - 1)$ .*

*Proof.* Let  $S \in \mathcal{T}(G, H)$  and  $L$  denote the set of left non-singular elements in  $S$ . Since  $[G : H] = n$  and  $|Itp(G, H)| = 1$ , then  $|L| < n$  otherwise by [6, Proposition 2.5],  $H$  is a normal subgroup of  $G$ , a contradiction. Hence  $|L| \leq n - 1$ . Let  $X$  be a set of principal isotopes of  $S$ . By the discussion in the last paragraph of section 2,  $|X| \leq n(n - 1)$ . Let  $S_1$  be an NRT of  $H$  in  $G$ . Since  $|Itp(G, H)| = 1$ ,  $S_1$  is isomorphic to some principal isotope of  $S$ . Hence  $|Iso(G, H)| \leq n(n - 1)$ .  $\square$

**Proposition 3.4.** *Let  $G$  be a finite group,  $H$  be a corefree subgroup of  $G$  such that  $|Itp(G, H)| = 1$ . Then there exists an isomorphism class  $\mathcal{A}$  of NRTs of  $H$  in  $G$  such that  $|\mathcal{A}| \geq \frac{m^{n-1}}{n(n-1)}$ , where  $|H| = m$ ,  $[G : H] = n$ .*

*Proof.* Since  $|H| = m$  and  $[G : H] = n$ ,  $|\mathcal{T}(G, H)| = m^{n-1}$ . By Proposition 3.3,  $|Iso(G, H)| \leq n(n - 1)$ . By Proposition 3.2,  $\langle S \rangle = G$  for all  $S \in \mathcal{T}(G, H)$ . Hence there exists isomorphism class  $\mathcal{A}$  of NRTs of  $H$  in  $G$  such that  $|\mathcal{A}| \geq \frac{m^{n-1}}{n(n-1)}$ .  $\square$

**Proposition 3.5.** *Let  $G$  be a finite group,  $H$  be a corefree subgroup of  $G$  such that  $|Itp(G, H)| = 1$ . Then  $|Aut_H(G)| \geq \frac{m^{n-1}}{n^2}$ , where  $|H| = m$ ,  $[G : H] = n$ .*

*Proof.* By Proposition 3.4, there exists an isomorphism class  $\mathcal{A}$  of NRTs of  $H$  in  $G$  such that  $|\mathcal{A}| \geq \frac{m^{n-1}}{n(n-1)} \geq \frac{m^{n-1}}{n^2}$ . By Proposition 3.2 and by Theorem 2.4,  $|Aut_H(G)| \geq |\mathcal{A}| \geq \frac{m^{n-1}}{n^2}$ .  $\square$

**Lemma 3.6.** *Let  $n \in \mathbf{N}$ ,  $n \geq 21$ . Then  $2^{n-3} \geq n^4$ .*

*Proof.* Assume that  $n = 21$ . Then  $2^{n-3} = 2^{18} = 2,62,144$  and  $n^4 = 21^4 = 1,94,481$ . Hence  $2^{n-3} \geq n^4$  for  $n = 21$ . Suppose that  $2^{k-3} \geq k^4$  for some  $k \in \mathbf{N}$ ,  $k > 21$ . Now  $2^{k+1-3} = 2 \cdot (2^{k-3})$  and since  $2^{k-3} \geq k^4$ , then  $2^{k+1-3} = 2 \cdot k^4 = k^4 + k^4 = k^4 + k \cdot k^3 \geq k^4 + 21 \cdot k^3 = k^4 + 7 \cdot k^3 + 5 \cdot k^3 + 7 \cdot k^3 + 2 \cdot k^3 \geq k^4 + 4 \cdot k^3 + 6 \cdot k^2 + 4 \cdot k + 1 = (k + 1)^4$ . Thus, by induction,  $2^{n-3} \geq n^4$  for all  $n \in \mathbf{N}$ ,  $n \geq 21$ .  $\square$

PROOF OF THE THEOREM 1.1: If possible, assume that  $H$  is not normal. Let  $K = \text{Core}_G(H)$ . Since  $|Itp(G, H)| = |Itp(G/K, H/K)|$  (see [5, Proposition 3.4]),  $G/K$  is solvable and  $H/K$  is corefree subgroup of  $G/K$ , we may assume that  $H$  is a corefree subgroup of  $G$ . Since  $G$  is a finite solvable group, there exists a non-trivial minimal normal subgroup  $N$  of  $G$ . Then by [10, 5.4.3],  $N$  is an elementary abelian  $p$ -group for some prime  $p$ . If  $N \cap H = \{1\}$ , then by [6, Proposition 3.1],  $|Itp(G, H)| > 1$ , a contradiction. Hence  $N \cap H \neq \{1\}$ . Since  $H$  is corefree subgroup of  $G$ ,  $N \cap H \neq N$ . Hence  $N \cap H$  is a non-trivial proper subgroup of  $N$ . Since  $N$  is an elementary abelian  $p$ -group, there exists a subgroup  $K'$  of  $N$  such that  $K' \in \mathcal{T}(N, H \cap N)$ . Hence  $K' \in \mathcal{T}(HN, H)$ . If  $HN = G$ , then by [6, Proposition 2.5],  $H$  is a normal subgroup of  $G$ , a contradiction. Hence  $HN \neq G$ . Let  $S_1 = K'L \in \mathcal{T}(G, H)$ , where  $L \in \mathcal{T}(G, NH)$ , and  $A_1$  be the middle-associators of  $S_1$ . Since  $K'$  is a subgroup of  $N$ , by [6, Proposition 2.4],  $K' \subseteq A_1$  and so  $|A_1| \geq |K'| > 1$ . By Proposition 3.1, this is a contradiction. Therefore  $H$  is a normal subgroup of  $G$ .  $\square$

PROOF OF THE THEOREM 1.2: If possible, assume that  $|Itp(G, H)| = 1$ . Since  $G$  is a finite non-abelian simple group,  $\text{Core}_G(H) = \{1\}$ . Let  $|H| = m$  and  $[G : H] = n$ . Then  $m \neq 1$  and  $n \neq 1, 2$ . Let  $S \in \mathcal{T}(G, H)$ . Then by Theorem 2.3 and Proposition 3.2,  $G = HS \cong G_S S \subseteq \text{Sym}(S \setminus \{1\})S$  and hence  $|G| \leq n!$ . Since  $G$  is a non-abelian simple group,  $|G| \geq 60$ . Hence  $n \geq 5$ .

Suppose that  $n = 5$ . Then  $m \geq 12$ . Since  $\text{Core}_G(H) = \{1\}$ , we can identify  $G$  with a subgroup of  $\text{Sym}(5)$ . Since  $G$  is a non-abelian simple group and  $|G| \geq 60$ ,  $G$  is isomorphic to  $A_5$ , alternating group of degree five. Hence  $m = 12$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{12^4}{5^2} > 144$ . But by [12, 2.17],  $|Aut(A_5)| = |Aut(G)| = 120$ , a contradiction. Hence  $n \geq 6$ .

Suppose that  $n = 6$ . Then  $m \geq 10$ . Let  $m = 10$  and so  $G \cong A_5$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{10^5}{6^2} > 1000$ . But by [12, 2.17],  $|Aut(A_5)| = |Aut(G)| = 120$ , a contradiction. Hence  $m \geq 11$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{m^5}{6^2} = \frac{m^2 \cdot m^3}{6^2} \geq \frac{m^2 \cdot 11^3}{6^2}$ . Since  $11^3 \geq 6^4$ ,  $|Aut(G)| \geq \frac{m^2 \cdot 6^4}{6^2} = m^2 \cdot 6^2 = |G|^2$ . By Theorem 2.5, this is a contradiction. Hence  $n \geq 7$ .

Let  $n = 7$ . Then  $m \geq 9$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq$

$\frac{m^6}{7^2} = \frac{m^2 \cdot m^4}{7^2} \geq \frac{m^2 \cdot 7^4}{7^2} = m^2 \cdot 7^2 = |G|^2$ . By Theorem 2.5, this is a contradiction. Hence  $n \geq 8$ .

Let  $n = 8$ . Then  $m \geq 8$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{m^7}{8^2} = \frac{m^2 \cdot m^5}{8^2} \geq \frac{m^2 \cdot 6^5}{8^2}$ . Since  $6^5 \geq 8^4$ ,  $|Aut(G)| \geq \frac{m^2 \cdot 8^4}{8^2} = m^2 \cdot 8^2 = |G|^2$ . By Theorem 2.5, we get this is a contradiction. Hence  $n \geq 9$ .

Let  $n = 9$ . Then  $m \geq 7$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{m^8}{9^2} = \frac{m^2 \cdot m^6}{9^2} \geq \frac{m^2 \cdot 5^6}{9^2}$ . Since  $5^6 \geq 9^4$ ,  $|Aut(G)| \geq \frac{m^2 \cdot 9^4}{9^2} = m^2 \cdot 9^2 = |G|^2$ . By Theorem 2.5, we get a contradiction. Hence  $n \geq 10$ .

Let  $10 \leq n \leq 15$ . Then  $m \geq 4$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{m^{n-1}}{n^2} = \frac{m^2 \cdot m^{n-3}}{n^2} \geq \frac{m^2 \cdot 4^{n-3}}{n^2}$ . Since  $4^{n-3} \geq n^4$ ,  $|Aut(G)| \geq \frac{m^2 \cdot n^4}{n^2} = m^2 \cdot n^2 = |G|^2$ . By Theorem 2.5, this is a contradiction. Hence  $n \geq 16$ .

Let  $16 \leq n \leq 29$ . Then  $m \geq 3$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{m^{n-1}}{n^2} = \frac{m^2 \cdot m^{n-3}}{n^2} \geq \frac{m^2 \cdot 3^{n-3}}{n^2}$ . Since  $3^{n-3} \geq n^4$ ,  $|Aut(G)| \geq \frac{m^2 \cdot n^4}{n^2} = m^2 \cdot n^2 = |G|^2$ . By Theorem 2.5, we get this is a contradiction. Hence  $n \geq 30$ .

Since  $n \geq 30$ ,  $m \geq 2$ . By Proposition 3.5,  $|Aut(G)| \geq |Aut_H(G)| \geq \frac{m^{n-1}}{n^2} = \frac{m^2 \cdot m^{n-3}}{n^2} \geq \frac{m^2 \cdot 2^{n-3}}{n^2}$ . By Lemma 3.6,  $|Aut(G)| \geq \frac{m^2 \cdot n^4}{n^2} = m^2 \cdot n^2 = |G|^2$ . By Theorem 2.5, we get a contradiction. This complete the proof.  $\square$

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Department of Mathematics,  
University of Allahabad,  
Prayagraj, 211002, India.  
E-mails: chandanrbsmath@gmail.com (C.S. Kharvar)  
ravindra@allduniv.ac.in (R.P. Shukla)