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Invo-regular clean rings

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Abstract. Let R be an associative ring with identity. In this paper, we introduce the notion of invo-regular clean rings where every element r can be written as $r = s + e$, where $s \in Inv_R(R)$ and $e \in Id(R)$. We study various properties of invo-regular clean rings. It is proved that, It is proved that, if R is a ring such that $2 \in U(R)$, then RG is invo-regular clean if and only if the group ring R is invo-regular clean.

1. Introduction

Let R be an associative ring with identity. An element v of R is said to be involution if $v^2 = 1$ [17]. Let $U(R)$, $Id(R)$, $Nil(R)$ and $Inv(R)$ will denote respectively the set of units, the set of idempotents, the set of nilpotents, the set of centrals and the set of involutions of R. The ring R is said to be clean if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$ [2, 18]. Generalizations of clean rings have been studied in [1, 3, 4, 16, 19, 23]. The ring R is said to be invo-clean if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v+e$ [5, 6]. Generalizations of invo-clean rings have been studied in [8, 9, 10, 11, 12, 13, 14, 20, 21, 22]. An element $s \in R$ is said to be an invo-regular element if there exist $v \in Inv(R)$ such that $s = svs$ [7]. Assume that $Inv_R(R)$ is the set of invo-regular elements of ring R.

In this paper, we introduce the notion of a invo-regular clean ring. Let R be a ring. Then an element $r \in R$ is said to be an invo-regular clean element if there exist $s \in Inv_R(R)$ and $e \in Id(R)$ such that $r = s + e$. A ring R is said to be invo-regular clean if each element in R is invo-regular clean. We study various properties of invo-regular clean rings as a proper generalization of invo-clean rings and a proper subclass of invo-regular rings. It is shown that, if R and S are two rings, $\phi : R \longrightarrow S$ is a ring epimorphism.

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If R is invo-regular clean, then S is invo-regular clean (Lemm 2.7). It is proved that, if R is a ring such that $2 \in U(R)$, then RG is invo-regular clean if and only if the group ring R is invo-regular clean (Lemma 2.18).

2. Main results

Definition 2.1. An element $s \in R$ is said to be an *invo-regular element* if there exist $v \in Inv(R)$ such that $s = svs$ [7]. Assume that $Inv_R(R)$ is the set of invo-regular elements of ring R.

Definition 2.2. An element $r \in R$ is said to be an *invo-regular clean element* if there exist $s \in Inv_R(R)$ and $e \in Id(R)$ such that $r = s + e$. A ring R is said to be invo-regular clean if each element in R is invo-regular clean.

It is clear that every invo-clean ring is an invo-regular clean ring. Since any idempotent is an invo-clean element, every idempotent is an invo-regular clean element. Simple examples of invo-regular clean rings that could be plainly verified are these: $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$. The following example shows that every invo-regular clean ring is not an invo-clean ring, in general.

Example 2.3. Let $R = \mathbb{Z}_5$. Then $Inv(R) = \{1, 4\}$, and $Inv_R(R) = \{1, 3, 4\}$, and $U(R) = \{1, 2, 3, 4\}$ and $Id(R) = \{0, 1\}$. Hence $R = \mathbb{Z}_5$ is an invo-regular clean ring which is not invo-clean.

Subsequently, we present fundamental properties of invo-regular clean rings.

Proposition 2.4. *Invo-regular rings are invo-regular clean.*

Proof. Invo-regular rings are invo-clen, by [7, Proposition 2.3]. Since invo-clean ring is an invo-regular clean ring, invo-regular rings are invo-regular clean. □

The following example shows that the converse of Proposition 2.7 does not hold.

Example 2.5. Let $R = \mathbb{Z}_5$. Then $Inv(R) = \{1, 4\}$, and $Inv_R(R) = \{1, 3, 4\}$, and $U(R) = \{1, 2, 3, 4\}$ and $Id(R) = \{0, 1\}$. Hence $R = \mathbb{Z}_5$ is an invo-regular clean ring which is not invo-regular.

Proposition 2.6. *Let R be a ring. Then R is invo-regular clean if and only if for any $r \in R$, there are $s \in Inv_R(R)$ and $e \in Id(R)$ such that $r = s - e$.*

Proof. Suppose that R is invo-regular clean. Then for any $r \in R$, there are $s \in Inv_R(R)$ and $e \in Id(R)$ such that $-r = s+e$. Hence $r = (-s)-e$ where clearly $-s \in Inv_R(R)$. Conversely, let $r \in R$ and choose $s \in Inv_R(R)$ and $e \in Id(R)$ such that $-r = s - e$. Then $r = (-s) + e$ is an invo-regular clean decomposition of r. □

Lemma 2.7. *Let R and S be two rings, $\phi : R \longrightarrow S$ be a ring epimorphism. If R isinvo-regular clean, then S is invo-regular clean.*

Proof. It is clear. □

Lemma 2.8. *Let I be a nil ideal of a ring R and R/I is invo-regular clean. Then R is invo-regular clean.*

Proof. Suppose that R/I is invo-regular clean where I is a nil ideal. Then idempotents lift modulo I and $I \subseteq J(R)$. Since $I \subseteq J(R)$, involution lift modulo I. Assume that $\overline{R} = R/I$ and $\overline{r} = r + I \in \overline{R}$ for $r \in R$. Let $\overline{s} = \overline{ev}$ be any invo-regular element in $\overline{R}$ where $\overline{v} \in Inv_R(\overline{R})$ and $\overline{e} \in Id(\overline{R})$. Then we may assume that $e \in Id(R)$ and $v \in Inv(R)$. It follows that $s = ev$ where $s = ev \in Inv_R(R)$. Thus, invo-regular elements of R are also lift modulo I. Therefore, r is a invo-regular clean element, and so R is invo-regular clean. □

Corollary 2.9. *Let R be a ring. Then R is invo-regular clean if and only if $R/Nil(R)$ is a invo-regular clean ring.*

Proof. Suppose that R is invo-regular clean and let $r+Nil(R) \in R/Nil(R)$. Write $r = (fv + e) + n$ for some $f, e \in Id(R)$, $v \in Inv(R)$ and $n \in Nil(R)$. Then $r+Nil(R) = fv+e+Nil(R) = (f+Nil(R))(v+Nil(R))+(e+Nil(R))$ where $(f + Nil(R)), (e + Nil(R)) \in Id(R/Nil(R))$ and $v + Nil(R) \in Inv(R/Nil(R))$. Therefore $R/Nil(R)$ is invo-regular clean Conversely, assume that $R/Nil(R)$ is an invo-regular clean ring. Since $Nil(R)$ is nil, R is invo-regular clean by Lemma 2.8. □

Let R be a ring. Then the prime radical $P(R)$ of R is nil. Therefore we have the following.

Corollary 2.10. *Let R be a ring. Then R is invo-regular clean if and only if $R/P(R)$ is a invo-regular clean ring.*

Proof. Follows from Lemmas 2.7 and 2.8. □

Remark 2.11. Let R be a ring and S be the ring consisting of all the following matrices

$$\begin{pmatrix} r & r_{1,2} & \cdots & r_{1,n} \\ 0 & r & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix}$$

such that $r, r_{i,j} \in R (i \lneqq j)$. Thus, it can be readily demonstrated that

$$I = \left\{ \begin{pmatrix} 0 & r_{1,2} & \cdots & r_{1,n} \\ 0 & 0 & \cdots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, r, r_{i,j} \in R (i \lneqq j) \right\}$$

constitutes an ideal of S that fulfills $I^n = 0$. Furthermore, it follows that $S/I \cong R$. Consequently, by applying Lemmas 2.7 and 2.8, we deduce that R is invo-regular clean if and only if S is invo-regular clean.

Corollary 2.12. *Let R be a ring. Then R is invo-regular clean if and only if $R[x]/\langle x^n \rangle$ is a invo-regular clean ring for all $n \geqslant 2$.*

Proof. Since

$$R[x]/\langle x^n \rangle \cong \left\{ \begin{pmatrix} r_1 & r_2 & r_3 \cdots & r_{n-1} & r_n & \\ 0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} \\ 0 & 0 & r_1 & \cdots & r_{n-3} & r_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r_1 & r_2 \\ 0 & 0 & 0 & \cdots & 0 & r_1 \end{pmatrix}, r_{i,j} \in R (i \lneqq j) \right\},$$

the conclusion follows directly from Remark 2.11. □

Given a ring R and an endomorphism σ of R, the skew polynomial ring $R[x, \sigma]$ consists of finite sums of the form $\sum_{i=0}^{n} a_i x^i$, where $a_i \in R$ for each i. The addition follows the usual rules, while multiplication is governed by the relation $xa = \sigma(a)x$ for all $a \in R$. Prior research has established that various properties can transfer between R and the quotient ring $R[x, \sigma]/\langle x^n \rangle$ for $n \in \mathbb{N}$. The following proposition demonstrates that the invo-regular clean property is among these transferable properties.

Proposition 2.13. *Let R be a ring, σ an endomorphism of R, and $n \in \mathbb{N}$. Then, R is invo-regular clean if and only if the quotient ring $R[x,\sigma]/\langle x^n \rangle$ is invo-regular clean.*

Proof. Consider the mapping $f : R[x,\sigma]/\langle x^n \rangle \longrightarrow R$ given by $f(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n)) = a_0$. It is evident that f is a ring epimorphism whose kernel corresponds to the ideal I of $R[x,\sigma]/\langle x^n \rangle$ generated by x. Furthermore, I is clearly a nil ideal. Consequently, the conclusion follows directly from Lemmas 2.7 and 2.8. □

Let R be a ring and M an R-module. The trivial ring extension of R by M, denoted $R \propto M$, is defined as the set $R \oplus M$ equipped with multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This structure forms a ring in which $0 \oplus M$ constitutes a nil ideal, and the quotient ring $(R \propto M)/(0 \oplus M)$ is isomorphic to R. By applying Lemmas 2.7 and 2.8, it follows that R is a invo-regular clean ring if and only if the trivial extension $R \propto M$ also satisfies the invo-regular clean property.

Now, consider a ring R and an R-bimodule W, which itself carries a ring structure possibly without a multiplicative identity subject to the following compatibility conditions for all $u, w \in W$ and $r \in R$: $(uw)r = u(wr)$, $(ur)w = u(rw)$, and $(ru)w = r(uw)$.

Under these conditions, the Dorroh extension of R by W, denoted $I(R,W)$, is defined as the abelian group $R \oplus W$ endowed with multiplication $(r,u)(a,w) = (ra, rw + ua + uw)$.

Proposition 2.14. *Let $I(R,W)$ denote the Dorroh extension of a ring R by a module W. If $I(R,W)$ possesses the invo-regular clean property, then R is invo-regular clean. Furthermore, if every $w \in W$ is idempotent and for every $e \in Id(R)$, $ew + we = 0$ the reverses holds.*

Proof. Suppose that $I(R,W)$ is invo-regular clean. Since R is a homomorphic image of $I(R,W)$ under the epimorphism $f : I(R,W) \longrightarrow R$ defined by $f(r,w) = r$, R is invo-regular clean by Lemma 2.7.

Conversely, assume that every $w \in W$ is idempotent and for every $e \in Id(R)$, $ew + we = 0$ and R is invo-regular clean. Let $(r,w) \in I(R,W)$ and write $r = s + e$ where $s \in Inv_R(R)$ and $e \in Id(R)$. Then $s = fv$ for some $v \in Inv(R)$ and $f \in Id(R)$ and so $(r,w) = (fv + e, w) = (fv, 0) + (e, w)$. Now, clearly $(fv, 0) = (f, 0)(v, 0)$ is a product of an idempotent and a involution in $I(R,W)$. Moreover, since $w \in W$ is idempotent and for every $e \in Id(R)$, $ew + we = 0$, then $(e,w) \in Id(I(R,W))$. Therefore, $I(R,W)$ is invo-regular clean. □

One can readily establish that whenever U constitutes a subring of W, the collection

$$I(R,U)=\{(r,u)\in I(R,W)\mid u\in U\}$$

forms a subring within $I(R,W)$. This observation directly yields the subsequent corollary to Lemma 2.7.

Corollary 2.15. *Suppose R and W are as defined in Proposition 2.14. If the nilradical $N(W)$ is itself a subring of W, then the ring $I(R,N(W))$ exhibits the invo-regular clean property if and only if R does.*

Let R be a unital ring and let S be a subring of R that shares the multiplicative identity with R . Define the following subset:

$$T[R,S]=\{(r_1,r_2,\cdots,r_n,s,s,\cdots)\mid r_i\in R, 1\leq i\leq n, s\in S, n\in\mathbb{N}\}.$$

It is straightforward to verify that $T[R,S]$, equipped with componentwise addition and multiplication, forms a ring with identity given by $(1,1,\cdots,1,1,1,\cdots)$. Moreover, the group of units and the set of nilpotent elements in this ring can be explicitly described as follows:

$$Inv_R(T[R,S])=\{(r_1,r_2,\cdots,r_n,s,s,\cdots)\mid r_i\in Inv_R(R), s\in Inv_R(S)\},$$

$$Id(T[R,S])=\{(r_1,r_2,\cdots,r_n,s,s,\cdots)\mid r_i\in Id(R), s\in Id(S)\}.$$

Consequently, the ensuing proposition follows directly from these observations.

Proposition 2.16. *Let R be a unital ring and S be a subring of R such that both share the same multiplicative identity. Then, the ring $T[R,S]$ possesses the invo-regular clean property if and only if both R and S individually satisfy the invo-regular clean condition.*

Lemma 2.17. *The (finite or infinite) direct product of invo-regular clean rings is again invo-regular clean.*

Proof. Standard technical checks are applied to this fact, and thus, we leave the detailed examination to the reader. □

If G is a group and R is a ring we denote the group ring over R by RG.

Lemma 2.18. *Let R be a ring such that $2\in U(R)$. Then RG is invo-regular clean if and only if R is invo-regular clean.*

Proof. Suppose RG is invo-regular clean. Since R is a homomorphic image of RG, R is invo-regular clean, by Lemma 2.7. Conversely, Since $2 \in U(R)$, $RG \cong R \times R$, by [15, Proposition 3]. Hence RG is invo-regular clean by Lemma 2.17. □

Recall that the ring of $n \times n$ upper triangular matrices over a ring R is denoted by $T_n(R)$. The next proposition establishes that if R is invo-regular clean, then $T_n(R)$ also inherits this property.

Proposition 2.19. *Let R be a ring. Then $T_n(R)$ is invo-regular clean if and only if R is invo-regular clean.*

Proof. Let I be the ideal of $T_n(R)$ consisting of all matrices with zero entries on the main diagonal. Clearly, I is a nil ideal. Furthermore, there is an isomorphism $T_n(R)/I \cong \prod_{i=1}^{n} R_i$, where $R_i = R$ for each $i \in \{1, 2, \cdots, n\}$. Since a finite direct product of invo-regular clean rings remains invo-regular clean by Lemma 2.17, the conclusion follows from Lemmas 2.7 and 2.8. □

Proposition 2.20. *Let R be a ring such that R is an invo-regular clean ring and $e \in Id(R)$. Then eRe is an invo-regular clean ring.*

Proof. It follows from [7, Proposition 2.16]. □

At this point, further technical details are required.

Lemma 2.21. *Let R be an invo-regular clean ring such that $Id(R) = \{0, 1\}$. Then $24 = 0$.*

Proof. Suppose that $3 \in R$. Since R is invo-regular clean, there exist $s \in Inv_R(R)$ and $e \in Id(R) = \{0, 1\}$ such that $3 = s + e$. If $e = 0$, then $3 = s = vf$ for some $v \in Inv(R)$ and $f \in Id(R) = \{0, 1\}$. Hence $3 = 0$ or $8 = 0$, and so $24 = 0$. If $e = 1$, then $2 = s + 1 = vf + 1$ for some $v \in Inv(R)$ and $f \in Id(R) = \{0, 1\}$. Hence $3 = vf + 1$, and so $2 = vf$. Hence $2 = 0$ or $3 = 0$, and so $24 = 0$. □

Proposition 2.22. *Let R be a ring such that $Id(R) = \{0, 1\}$. Then R is invo-regular clean if and only if $R = R_1 \times R_2$, where R_1 is an invo-regular clean ring of characteristic 3 and R_2 is an invo-regular clean ring of characteristic 8.*

Proof. Since R is invo-regular clean, $24 = 0$, by Lemma 2.21. On the other hand $(3, 8) = 1$. Hence $R = 3R \oplus 8R$. Then $R/3R \cong 8R$ and $R/8R \cong 3R$ which both imply that $R \cong R_1 \times R_2$ where $R_1 = R/3R$ and $R_2 = R/8R$. But homomorphic images of an invo-regular clean ring are again invo-regular clean rings, by Lemma 2.7, as required. □

We proceed by examining a distinctive property within the class of invo-regular clean rings, characterized as follows.

Definition 2.23. A ring R is said to be *uniquely invo-regular clean* if, for every $0 \neq r \in R$, there are a unique $s \in Inv_R(R)$ and a unique idempotent $e \in Id(R)$ such that $r = s + e$.

Proposition 2.24. *Let R be a ring. Then evey central idempotents is uniquely invo-regular clean.*

Proof. Let $e \in Id(R)$ is central. Then we have $e = (1 - e) + (1 - 2e)$. Assume that $e = s + f$ such taht $s \in Inv_R(R)$ and $f \in Id(R)$. Since $es = se$, $s + f = (s + f)^2 = s^2 + 2fs + f$. Hence $1 = s + 2f$, and so $s = 1 - 2f$. Therefore $f = 1 - e$ and $s = 1 - 2e \in Inv_R(R)$. □

Corollary 2.25. *Let R be a Boolean ring. Then R is uniquely invo-regular clean.*

Proof. Follows from Proposition 2.24. □

Lemma 2.26. *Let R be a ring. Then R is uniquely invo-regular clean if and only if, for each $0 \neq r = s + e \in R$, such taht $s \in Inv_R(R)$ and $e \in Id(R)$ is a unique and there is a unique $v \in Inv(R)$ with $s = vf$ for some $f \in Id(R)$.*

Proof. Suppose that R is uniquely invo-regular clean and $0 \neq rR$. Hence there exist a unique $s \in Inv_R(R)$ and a unique $e \in Id(R)$. Then there is a unique $v \in Inv(R)$ with $s = vf$ for some $f \in Id(R)$, by [7, Proposition 2.9], as required. Conversly, asuume taht $r = s + e$ such that there is a unique $v \in Inv(R)$ with $s = vf$ for some $f \in Id(R)$ and $e \in Id(R)$ is unique. Therefore $s \in Inv_R(R)$ is unique, and so R is uniquely invo-regular clean. □

We close the article with the following two problems.

Problem 1. *What is the behaviour of the matrix rings over invo-regular clean rings?*

Problem 2. *Is it true that for a non-zero commutative ring R with identity 1, the ring $R[x]$ can never be invo-regular clean?*

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References

[1] **M.Y. Ahn, D.D. Anderson**, *Weakly clean rings and almost clean rings*, Rocky Mountain J. Math. **36** (2006), 783 – 799.

[2] **D.D. Anderson, V.P. Camillo** *Commutative rings whose elements are a sum of a unit and an idempotent*, Commun. Algebra **30** (2002), 3327 – 3336.

[3] **A.Y.M. Chin, K.T. Qua** *A note on weakly clean rings*, Acta Math. Hungar. **132** (2001), 113 – 116.

[4] **P.V. Danchev**, *On weakly clean and weakly exchange rings having the strong property*, Publ. L'Institut Math. **101** (2017), 135 – 142.

[5] **P.V. Danchev**, *Invo-clean unital rings*, Commun. Korean Math. Soc. **32** (2017), 19 – 27.

[6] **P.V. Danchev**, *Corners of invo-clean unital rings*, Pure Math. Sci. **7** (2018), 27 – 31.

[7] **P.V. Danchev**, *Invo-regular unital rings*, Annales Univ. Mariae Curie-Sklodowska, Sec. A–Mathematica **72** (2018). 45 – 53.

[8] **P.V. Danchev**, *Commutative invo-clean group rings*, Univers. J. Math. & Math. Sci. **11** (2018), 1 – 6.

[9] **P.V. Danchev**, *Quasi invo-regular rings*, J. Prime Res. Math. **15** (2019), 1 – 4.

[10] **P.V. Danchev**, *Commutative weakly invo-clean group rings*, Ural Math. J. **5** (2019), 48 – 52.

[11] **P.V. Danchev**, *Commutative feebly invo-clean group rings*, Vestnik Tomsk Gos. Univ., Ser. Mat. & Mekh. **61** (2019), 5 – 10.

[12] **P.V. Danchev**, *Locally invo-regular rings*, Azerbaijan J. Math. **11** (2021), 28 – 44.

[13] **P.V. Danchev**, *Quasi invo-clean rings*, Bull. Transilvania Univ. Brasov - Series III (Math. & Comp. Sci.) **63** (2021), 71 – 80.

[14] **P.V. Danchev**, *Weakly invo-clean rings having weak involution*, Vestnik Udmurtskogo Univ. – Matematika, Mekhanika, Komp'yuternye Nauki **32** (2022), 18 – 25.

[15] **J. Han, W.K. Nicholson**, *Extensions of clean rings*, Commun. Algebra **29** (2001), 2589˘2595.

[16] **T. Kosan, S. Sahinkaya, Y. Zhou**, *On weakly clean rings*, Commun. Algebra **45** (2017), 3494 – 3502.

[17] **T.Y. Lam**, *A First Course in Noncommutative Rings*, Graduate Texts in Math., Vol. 131, Springer, (2001).

[18] **W.K. Nicholson** *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269 – 278.

[19] **F. Rashedi**, *On weakly f-clean rings*, Publ. L'Institut Math. Nouvelle Série, **112** (2022), 53 – 57.

[20] **F. Rashedi**, Weakly quasi invo-clean rings, Quasigroups and Related Systems **31** (2023), 117 – 124.

[21] **F. Rashedi**, *Weakly $g(x)$-invo-clean rings*, Quasigroups and Related Systems **31** (2023), 285 – 292.

[22] **F. Rashedi**, *Weakly $g(x)$-quasi invo-clean rings*, Transactions of A. Razmadze Math. Inst. **178** (2024), 111 – 114.

[23] **F. Rashedi**, *On weakly f-clean rings*, Quasigroups and Related Systems **32** (2024), 109 – 118.

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