Quasigroups and Related Systems **34** (2026), 163 – 176

https://doi.org/10.56415/qrs.v34.13

Annihilator graph of a commutative semigroup whose zero divisor graph is a star graph

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Abstract. Let S be a commutative semigroup with zero and $\Gamma(S)$ be the zero divisor graph of S which is a star graph with center x and n end vertices. Also, let $Z^*(S)$ be the set of all nonzero zero divisors of S. The annihilator graph for a commutative semigroup S, which is denoted by $AG(S)$, is an undirected graph with vertex set $Z^*(S)$ and two distinct vertices a and b are adjacent if and only if $\mathrm{ann_S}(ab) \neq \mathrm{ann_S}(a) \cup \mathrm{ann_S}(b)$, where $\mathrm{ann_S}(a) = \{s \in S \mid as = 0\}$. In this paper, we study the annihilator graph associated with a commutative semigroup with zero, when $\Gamma(S)$ is a star graph. We prove that if $Z(S) \neq S$, then $AG(S)$ is a star graph or a complete graph, and if If $Z(S) = S$, then $AG(S)$ is an empty graph or $AG(S) \cong K_1 \cup K_n$. Moreover, we show that if $AG(S)$ is a star graph with center x, then $Z(S) \neq S$ and $\Gamma(S)$ is a star graph with center x and the product of all end vertices of $\Gamma(S)$ is not equal to the center vertex.

1. Introduction

In this paper, we assume that S is a commutative semigroup with zero whose operation is written multiplicatively and $Z(S)$ is the set of all zero divisors of S. Also $Z^*(S) = Z(S) \setminus \{0\}$.

For any commutative semigroup S with zero element 0, $\Gamma(S)$ is the zero divisor graph of S with vertex set $Z^*(S)$ and for each two distinct vertices u and v in $Z^*(S)$, u is adjacent to v in $\Gamma(S)$ if and only if $uv = 0$. $\Gamma(S)$ is a connected graph and its diameter is less than or equal to three. For more results on the zero divisor graphs one can see [5, 7, 9, 10, 11, 12, 13].

In [6], A. Badawi introduced the concept of the annihilator graph for a commutative ring R, denoted by $AG(R)$, with vertices $Z^*(R)$ and two distinct vertices a and b are adjacent in $AG(R)$ if and only if $\mathrm{ann_R}(ab) \neq$

2010 Mathematics Subject Classification: 05C25, 20M14.

Keywords: Commutative semigroup, zero divisor graph, annihilator graph, star graph.

$\mathrm{ann_R}(a) \cup \mathrm{ann_R}(b)$, where $\mathrm{ann_R}(a) = \{r \in R \mid ar = 0\}$. It was proved that $AG(R)$ is connected with $\mathrm{diam}(AG(R)) \leqslant 2$ and $\mathrm{gr}(AG(R)) \leqslant 4$.

In [1], the annihilator graph for a commutative semigroup S, which is denoted by $AG(S)$, was introduced and studied. The graph $AG(S)$ is an undirected graph with vertex set $Z^*(S)$ and two distinct vertices a and b are adjacent if and only if $\mathrm{ann_S}(ab) \neq \mathrm{ann_S}(a) \cup \mathrm{ann_S}(b)$, where $\mathrm{ann_S}(a) = \{s \in S \mid as = 0\}$. Some basic properties of $AG(S)$ were investigated in [1]. For example, it is proved that if $Z(S) \neq S$, then $\Gamma(S)$ is a subgraph of $AG(S)$, and so $AG(S)$ is connected. Also if $Z(S) = S$, then $AG(S)$ may be connected or disconnected, and if there exists $a \in S^* = S \setminus \{0\}$ such that a is adjacent to all vertices in $\Gamma(S)$, then a is an isolated vertex in $AG(S)$. Also it is showed that if $\Gamma(S)$ is a star graph and $Z(S) \neq S$, then $AG(S)$ is a refinement of a star graph and if $Z(S) = S$, then the annihilator graph of S is a disconnected graph. In [1] and [2], all semigroups S whose annihilator graphs have three and four vertices are characterized. In [3], the structure of the annihilator graph of a commutative semigroup S whose $\Gamma(S)$ is a refinement of a star graph is studied. Also in [4], the annihilator graph associated with a commutative semigroup with zero is studied by using the zero divisor graph $\Gamma(S)$, where $\Gamma(S)$ is a complete graph K_n with an end vertex $u \notin \mathrm{V}(K_n)$ and u is only adjacent to $z \in \mathrm{V}(K_n)$ and it is showed that if $Z(S) = S$, then the annihilator graph of S is a disconnected graph.

In this paper, we study the structure of the annihilator graph of S when $\Gamma(S)$ is a star graph with center x and n end vertices. We show that if there are two end vertices y and z in $\Gamma(S)$ such that $yz = x$, then in the case that $Z(S) \neq S$, we have $AG(S)$ is a complete graph K_{n+1}, and if $Z(S) = S$, then $AG(S) \cong K_1 \cup K_n$, where x is an isolated vertex. Also, we prove that if for all end vertices y and z in $\Gamma(S)$, $yz \neq x$, then, in the case that $Z(S) \neq S$, we have $AG(S)$ is a star graph with center x, and if $Z(S) = S$, then $AG(S)$ is isomorphic to $(n+1)K_1$. Moreover, we show that if $AG(S)$ is a star graph with center x, then $Z(S) \neq S$ and $\Gamma(S)$ is a star graph with center x and the product of all end vertices of $\Gamma(S)$ is not equal to the center vertex.

2. Preliminaries

In this section, first we recall some definitions and notations of graphs. We use the standard terminology of graphs following [8].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If a is adjacent to b in G, then the edge between a and b is denoted by $\{ab\}$ and we write $a \sim b$.

Let H and G be two graphs such that $V(G) \cap V(H) = \emptyset$ and $E(G) \cap E(H) = \emptyset$. The union of the graphs H and G, which is denoted by $H \cup G$, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(H) \cup E(G)$. If there exists a path between any two distinct vertices of G, then G is a connected graph, and if for each two distinct vertices x and y we have x is adjacent to y, then G is a complete graph. The complete graph with n vertices is denoted by K_n. If no two vertices of G are adjacent, then G is an empty graph and nK_1 is the empty graph with n vertices, where mK_n is the union of m copies of the complete graph K_n.

We say that u is an end vertex in G, if u is adjacent to only one vertex of G. Also if for each vertex $x \in V(G)$ we have u is not adjacent to x, then u is an isolated vertex in G. Suppose that H and G are two graphs. We use the notation $G \leqslant H$ to denote that G is a subgraph of H and if H is isomorphic to G, we write $H \cong G$. Also $G \setminus \{\{x_1y_1\}, \{x_2y_2\}, \{x_3y_3\}, \dots, \{x_ny_n\}\}$ is a graph such that the edges $\{x_1y_1\}, \{x_2y_2\}, \{x_3y_3\}, \dots, \{x_ny_n\}$ are deleted.

A graph G with $n+1$ vertices is called a star graph, and is denoted by $K_{1,n}$, if there exists a vertex $x \in V(G)$ such that $\mathrm{d}(x) = n$, and for each vertex $y \in V(G) \setminus \{x\}$, we have $\mathrm{d}(y) = 1$. The vertex x is called the center of $K_{1,n}$. Also P_n is the path of length n.

In the rest of this section, we state some results about the structure of the graph $AG(S)$ in the case that $\Gamma(S)$ is a star graph with $|Z^*(S)| \leqslant 3$.

If $\Gamma(S) \cong K_1$, then $AG(S) \cong K_1$. Let $\Gamma(S) \cong K_{1,1} \cong K_2$. Clearly if $Z(S) \neq S$, then $AG(S) \cong K_{1,1} \cong K_2$, and if $Z(S) = S$, then $AG(S) \cong 2K_1$.

Now, suppose that $\Gamma(S)$ is a star graph $K_{1,2}$ with center x and end vertices y and z, and $Z(S) \neq S$. In [1, Section 4], it was showed that if $yz = x$ and $y^2 = z^2 = 0$ or $yz = y$ or $yz = z$, then $AG(S) \cong K_{1,2}$ and if $yz = x$ and $y^2 \neq 0$ or $z^2 \neq 0$, then $AG(S) \cong K_3$. Also, it was proved that if $Z(S) = S$ and $yz = x$ and $y^2 = z^2 = 0$ or $yz = y$ or $yz = z$, then $AG(S) \cong 3K_1$ and if $yz = x$ and $y^2 \neq 0$ or $z^2 \neq 0$, then $AG(S) \cong K_2 \cup K_1$, with x is an isolated vertex.

In [2], all semigroups S such that their annihilator graphs have four vertices were characterized. Moreover, a complete (up to isomorphism) description of all pairs (G, S), where G is a graph and S is a commutative semigroup such that G is the annihilator graph of S and G has four vertices, was given. Now suppose that $\Gamma(S)$ is a star graph $K_{1,3}$ with center x and end vertices y, z and w, and $Z(S) \neq S$. In [2], it was showed that if $yz \neq x$ and $yw \neq x$ and $wz \neq x$, then $AG(S) \cong K_{1,3}$ with center x, and if $yz = x$

or $yw = x$ or $wz = x$, then $AG(S) \cong K_3$. Also, it was proved that if $Z(S) = S$, $yz \neq x$, $yw \neq x$ and $wz \neq x$, then $AG(S) \cong 3K_1$, and if $yz = x$ or $yw = x$ or $wz = x$, then $AG(S) \cong K_3 \cup K_1$, where x is an isolated vertex.

3. The structure of $AG(S)$

In this section, we study the annihilator graph of a commutative semigroup S with $|Z^*(S)| \geqslant 4$, whose $\Gamma(S)$ is a star graph. To do this, we assume that $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, u_3, \dots, u_n\}$, where $n \geqslant 3$. Therefore for all $u_i \neq u_j$, we have $xu_i = 0$ and $u_iu_j \neq 0$, and so $u_iu_j = x$ or there is $u_t \in U$ such that $u_iu_j = u_t$. Also $x^2 = 0$ or $x^2 = x$.

For each two end vertices of $\Gamma(S)$ we have one of the following statements.

(i) The product of all end vertices of the zero divisor graph is equal to the center vertex, that is, $u_iu_j = x$, for all $u_i, u_j \in U$ and $u_i \neq u_j$.

(ii) The product of all end vertices of the zero divisor graph is not equal to the center vertex, that is, $u_iu_j \neq x$, for all $u_i, u_j \in U$ and $u_i \neq u_j$.

(iii) The product of some end vertices of the zero divisor graph is equal to the center vertex and the product of some other is not equal to the center vertex, that is, there are $u_i, u_j, u_k, u_t \in U$ such that $u_i \neq u_j$, $u_t \neq u_k$, $u_iu_j = x$ and $u_ku_t \neq x$.

First, we examine the Case (i) where the product of all end vertices of the zero divisor graph is equal to the center vertex.

Lemma 3.1. *Suppose that that for all $u_i, u_j \in U$ and $u_i \neq u_j$, we have $u_iu_j = x$. Then $x^2 = 0$ and $u_i^2 \in \{0, x\}$.*

Proof. Since $u_iu_j = x$, we have $x^2 = x(u_iu_j) = (xu_i)u_j = 0u_j = 0$. For all $u_i, u_j \in U$ and $u_i \neq u_j$, we have $u_iu_j = x$. Hence $u_i^2u_j = u_i(u_iu_j) = u_ix = 0$ and so $u_i^2 \in \mathrm{ann}_S(u_j) \subseteq \{0, x, u_j\}$, for all $j \neq i$. If $u_i^2 = u_j$, then since $n \geqslant 3$, there is $k \notin \{i, j\}$ such that $u_iu_k = u_ju_k = x$. Thus $x = u_ju_k = u_i^2u_k = u_i(u_iu_k) = u_ix = 0$, which is impossible. Therefore $u_i^2 \in \{0, x\}$. □

Lemma 3.2. *If $u_iu_j = x$, for all $u_i, u_j \in U$ and $i \neq j$, then u_i is adjacent to u_j in $AG(S)$.*

Proof. By Lemma 3.1, we have $x^2 = 0$. Also for all i, we have $u_i x = 0$. Thus $\text{ann}_S(x) = Z(S)$. Since $n \geqslant 3$, there is $k \notin \{i, j\}$ that $u_i u_k \neq 0$ and $u_j u_k \neq 0$, and so $u_k \notin \text{ann}_S(u_i) \cup \text{ann}_S(u_j)$ which implies that $\text{ann}_S(u_i) \cup \text{ann}_S(u_j) \neq Z(S) = \text{ann}_S(x) = \text{ann}_S(u_i u_j)$. Therefore u_i is adjacent to u_j in $AG(S)$, for all $i \neq j$. □

By the above lemma we have the following theorem.

Theorem 3.3. *Suppose that $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, u_3, \ldots, u_n\}$, where $n \geqslant 3$. Also assum that for all $i \neq j$, $u_i u_j = x$. Then one of the following statements holds.*

(*i*) *If $Z(S) \neq S$, then $AG(S) \cong K_{n+1}$.*

(*ii*) *If $Z(S) = S$, then $AG(S) \cong K_1 \cup K_n$, where x is an isolated vertex.*

Proof. If $Z(S) \neq S$, then by [1, Theorem 3.1], we have $\Gamma(S) \leqslant AG(S)$, and if $Z(S) = S$, then by [1, Theorem 3.8], we have x is an isolated vertex in $AG(S)$. Now, by Lemma 3.2, the results hold. □

In [1, Theorem 4.1], we have showed that if $Z(S) = \{0, x, y, z\} \neq S$ and $xy = xz = 0$, then $AG(S) \cong K_{1,2}$ with center x if and only if one of the following statements holds:

(*i*) $yz = y$ or $yz = z$.

(*ii*) $yz = x$ and $x^2 = y^2 = z^2 = 0$.

In this case $\Gamma(S)$ is a star graph $K_{1,2}$ with center x and end vertices $U = \{y, z\}$. In (ii) we have $yz = x$ but $AG(S) \cong K_{1,2}$, and so $AG(S)$ is not a complete graph K_3. Therefore the condition that $n \geqslant 3$ in Theorem 3.3 is necessary.

Also in [1, Theorem 4.3], we have proved that if $Z(S) = \{0, x, y, z\} = S$ and $xy = xz = 0$, then $AG(S) \cong K_2 \cup K_1$ with x is an isolated vertex if and only if $zy = x$ and either we have $y^2 \neq 0$ or $z^2 \neq 0$. Otherwise $AG(S) \cong 3K_1$.

Moreover, Theorem 3.3 shows that if $AG(S)$ is a complete graph, then $\Gamma(S)$ can be a star graph, but in [1, Theorem 4.2], it was showed that if $Z(S) = \{0, x, y, z\} \neq S$ and $xy = xz = 0$, then $AG(S) \cong K_3$ if and only if one of the following statements holds:

(*i*) $yz = 0$.

(*ii*) $yz = x$ and we have either $y^2 \neq 0$ or $z^2 \neq 0$.

Therefore if $Z(S) = \{0, x, y, z\} \neq S$, then $AG(S) \cong K_3$ if and only if $\Gamma(S) \cong K_3$.

Example 3.4. Suppose that S is a commutative semigroup with zero and $Z^*(S) = \{w, x, y, z\}$. Also assume that $xy = xz = xw = 0$, $wy = wz = yz = x$, $x^2 = 0$ and $y^2, z^2, w^2 \in \{0, x\}$. In this case $\Gamma(S)$ is a star graph $K_{1,3}$ with center x and end vertices $U = \{w, y, z\}$. If $Z(S) \neq S$, then by Theorem 3.3, we have $AG(S) \cong K_4$, and if $Z(S) = S$, then by Theorem 3.3, we have $AG(S) \cong K_1 \cup K_3$. The zero divisor graph and the annihilator graph are pictured in Figure 1.

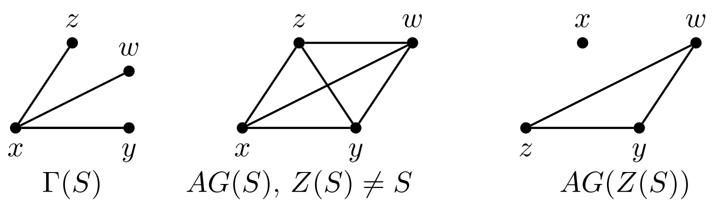


Figure 1

Example 3.5. Let $S = \{\bar{0}, \bar{1}, \bar{4}, \bar{12}, \bar{16}, \bar{20}, \bar{28}\}$ be a semigroup with multiplicative operation modulo 32. Then $\Gamma(S)$ is a star graph with center $\bar{16}$ and end vertices $U = \{\bar{4}, \bar{12}, \bar{20}, \bar{28}\}$. Also $\bar{4}.\bar{12} = \bar{4}.\bar{20} = \bar{4}.\bar{28} = \bar{12}.\bar{20} = \bar{12}.\bar{28} = \bar{20}.\bar{28} = \bar{16}$. Then $AG(S)$ is a complete graph K_5 and $AG(Z(S)) \cong K_1 \cup K_4$, with $\bar{16}$ is an isolated vertex.

Now, we study the case when the product of each two distinct end vertices of the zero divisor graph is not equal to the center vertex.

Lemma 3.6. *Assume that* $\Gamma(S)$ *is a star graph with center* x *and end vertices* $U = \{u_1, u_2, u_3, \ldots, u_n\}$. *Also assum that for all* $i \neq j$, *we have* $u_i u_j \neq x$. *Then we have* $u_i u_j = u_i$ *or* $u_i u_j = u_j$, *or there is* $t \notin \{i, j\}$ *such that* $u_i u_j = u_t$ *and* $u_t^2 \neq 0$.

Proof. Suppose that $u_i u_j \notin \{u_i, u_j\}$. Since $u_i u_j \notin \{x, 0\}$, there is $t \notin \{i, j\}$ such that $u_i u_j = u_t$, and we have $u_i u_j^2 = (u_i u_j) u_j = u_t u_j \neq 0$. Thus $u_j^2 \neq 0$, and so $\mathrm{ann}_S(u_j) = \{0, x\}$. Since $u_i u_j = u_t$, we have $u_t^2 = u_t(u_i u_j) = (u_t u_i) u_j$. Now, if $u_t^2 = 0$, then $(u_t u_i) u_j = 0$, and hence $(u_t u_i) \in \mathrm{ann}_S(u_j) = \{0, x\}$. Therefore $(u_t u_i) = 0$ or $(u_t u_i) = x$, which is impossible. Consequently $u_t^2 \neq 0$. □

Lemma 3.7. *Suppose that* $\Gamma(S)$ *is a star graph with center* x *and end vertices* $U = \{u_1, u_2, u_3, \dots, u_n\}$. *Also assum that for all* $i, j \in \{1, 2, 3, \dots, n\}$, *we have* $u_i u_j \neq x$. *Then* u_i *is not adjacent to* u_j *in* $AG(S)$, *for all* $i, j \in \{1, 2, 3, \dots, n\}$.

Proof. Since $u_i u_j \neq x$, by Lemma 3.6, $u_i u_j = u_i$ or $u_i u_j = u_j$ or there is $t \notin \{i, j\}$ such that $u_i u_j = u_t$ and $u_t^2 \neq 0$. If $u_i u_j = u_i$ or $u_i u_j = u_j$, then u_i is not adjacent to u_j in $AG(S)$, for all $i, j \in \{1, 2, 3, \dots, n\}$. If $u_i u_j = u_t$, then $u_t^2 \neq 0$. Since $u_i u_j = u_t$, we have $u_i^2 u_j = u_i(u_i u_j) = u_i u_t \neq 0$ and so $u_i^2 \neq 0$. Also $u_j^2 \neq 0$. Hence $\mathrm{ann}_S(u_i) = \mathrm{ann}_S(u_j) = \mathrm{ann}_S(u_t) = \{0, x\}$, and so $\mathrm{ann}_S(u_i) \cup \mathrm{ann}_S(u_j) = \mathrm{ann}_S(u_t) = \mathrm{ann}_S(u_i u_j)$. Therefore u_i is not adjacent to u_j in $AG(S)$, for all $i, j \in \{1, 2, 3, \dots, n\}$. □

Theorem 3.8. *Suppose that* $\Gamma(S)$ *is a star graph with center* x *and end vertices* $U = \{u_1, u_2, \dots, u_n\}$. *Also assum that for all* $i, j \in \{1, 2, 3, \dots, n\}$, *we have* $u_i u_j \neq x$. *Then one of the following statements holds.*

(*i*) *If* $Z(S) \neq S$, *then* $AG(S) \cong K_{1,n}$ *with center* x.

(*ii*) *If* $Z(S) = S$, *then* $AG(S) \cong (n+1)K_1$.

Proof. If $Z(S) \neq S$, then by [1, Theorem 3.1], we have $\Gamma(S) \leqslant AG(S)$, and if $Z(S) = S$, then by [1, Theorem 3.8], we have x is an isolated vertex in $AG(S)$. Now, by Lemma 3.7, the results hold. □

Example 3.9. Suppose that S is a commutative semigroup with zero and $Z^*(S) = \{w, x, y, z\}$ and $xy = xz = xw = 0$. Also assume that $wy \neq x$, $wz \neq x$ and $yz \neq x$. Then one of the following four statements holds.

(*i*) $wy = wz = zy = z$, $z^2 = z$, $x^2 \in \{0, x\}$ and we have one of the following three statements.

(1) $y^2 = y$ and $w^2 \in \{w, z\}$.
(2) $y^2 = w$ and $w^2 = z$.
(3) $y^2 = z$ and $w^2 \in \{w, y, z\}$.

(*ii*) $wy = yz = y$, $wz = z$, $x^2 \in \{0, x\}$ and we have one of the following four statements.

(1) $y^2 = 0$, $w^2 = w$ and $z^2 \in \{w, z\}$.
(2) $y^2 = y$, $w^2 = w$ and $z^2 \in \{w, y, z\}$.

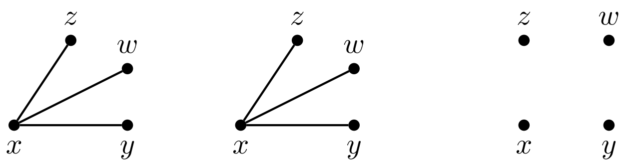
(3) $y^2 = z^2 = z$ and $w^2 = w$.

(4) $w^2 = z^2 = z$ and $y^2 \in \{0, y, z\}$

(iii) $wy = z$, $wz = z$, $yz = w$, $x^2 \in \{0, x\}$ and $w^2 = y^2 = z^2 = w$.

(iv) $wy = y$, $wz = z$, $yz = w$, $x^2 \in \{0, x\}$, $w^2 = w$, $z^2 = y$ and $y^2 = z$.

The graph $\Gamma(S)$ is a star graph $K_{1,3}$ with center x and end vertices $U = \{w, y, z\}$. If $Z(S) \neq S$, then by Theorem 3.8, we have $AG(S) \cong K_{1,3}$, and if $Z(S) = S$, then by Theorem 3.8, we have $AG(S) \cong 4K_1$. The zero divisor graph and the annihilator graph are pictured in Figure 2.



$\Gamma(S)$ $AG(S)$, $Z(S) \neq S$ $AG(Z(S))$

Figure 2

Example 3.10. Let $S = \{\bar{0}, \bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{6}, \bar{8}\}$ be a semigroup with multiplicative operation modulo 10. Then $\Gamma(S)$ is a star graph with center $\bar{5}$ and end vertices $U = \{\bar{2}, \bar{4}, \bar{6}, \bar{8}\}$. Also $\bar{2}.\bar{4} = \bar{6}.\bar{8} = \bar{8}$, $\bar{2}.\bar{6} = \bar{4}.\bar{8} = \bar{2}$, $\bar{2}.\bar{8} = \bar{6}$ and $\bar{6}.\bar{4} = \bar{4}$. Then $AG(S)$ is a star graph with center $\bar{5}$ and end vertices $U = \{\bar{2}, \bar{4}, \bar{6}, \bar{8}\}$, and $AG(Z(S))$ is an empty graph $5K_1$.

Finally, we study the case when the product of some end vertices of the zero divisor graph is equal to the center vertex and the product of some other is not equal to the center vertex.

Lemma 3.11. *Suppose that $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \ldots, u_n\}$. Also assum that $u_1u_2 = x$ and there is $k \in \{3, 4, \ldots, n\}$ such that $u_1u_k \neq x$. Then the following statements hold.*

(i) *$u_1u_i \neq u_1$ and $u_2u_i \neq u_2$, for all $i = 1, 2, \ldots, n$.*

(ii) *$u_1^2 = u_2$ and $u_2^2 = 0$.*

(iii) *$u_2u_i = x$, for all $i \neq 2$.*

(iv) *$u_1u_i = u_2$, for all $i \neq 2$.*

(*v*) $u_iu_j = u_2$, *for all* $i, j \neq 2$.

Proof. (*i*). Suppose that there exists $1 \leqslant i \leqslant n$ such that $u_1u_i = u_1$. We have $x = u_2u_1 = u_2(u_1u_i) = (u_2u_1)u_i = xu_i = 0$, which is impossible. Thus $u_1u_i \neq u_1$. Also $u_2u_i \neq u_2$, for all $i = 1, 2, \ldots, n$.

(*ii*). Since there is $k \in \{3, \ldots, n\}$ such that $u_1u_k \neq x$, we have $u_1u_k = u_t$ and by (i) $t \neq 1$. Thus $u_1^2u_k = u_1u_t \neq 0$ and hence $u_1^2 \notin \mathrm{ann}_S(u_k) \supseteq \{0, x\}$. Therefore $u_1^2 \neq 0, x$. On the other hand, $u_1^2u_2 = u_1(u_1u_2) = u_1x = 0$, and so $u_1^2 \in \mathrm{ann}_S(u_2) \subseteq \{0, x, u_2\}$. Thus $u_1^2 = u_2$. Since $u_1^2 = u_2$, we have $u_2^2 = u_2u_1^2 = (u_2u_1)u_1 = xu_1 = 0$, and so $u_2^2 = 0$

(*iii*). By (*i*), $u_2u_i \neq u_2$, for all $i = 1, 2, 3, \ldots, n$. If $u_2u_i = u_1$, then by (*ii*), we have $u_2 = u_1^2 = u_1(u_2u_i) = (u_1u_2)u_i = xu_i = 0$, which is impossible and so $u_2u_i \neq u_1, u_2$.

If there is $t \in \{3, \ldots, n\}$ such that $u_2u_i = u_t$, then $u_1u_t = u_1(u_2u_i) = (u_1u_2)u_i = xu_i = 0$, which is impossible. Therefore $u_2u_i \neq u_t$, for all $t = 1, 2, 3, \ldots, n$. Also for $i \neq 2$, we have $u_2u_i \neq 0$. Therefore $u_2u_i = x$, for all $i \neq 2$.

(*iv*). Since $u_1u_k \notin \{x, 0\}$, there is t such that $u_1u_k = u_t$, and so $0 = xu_k = (u_1u_2)u_k = u_2(u_1u_k) = u_2u_t$. Thus $u_t = u_2$, and hence $u_1u_k = u_2$.

If $u_1u_i = x$ and $i \neq 2$, then $0 = xu_k = (u_1u_i)u_k = u_i(u_1u_k) = u_iu_2$, which is impossible, and so $u_1u_i \notin \{x, 0\}$. Therefore there is t such that $u_1u_i = u_t$, and so $(u_2u_t) = u_2(u_1u_i) = (u_2u_1)u_i = xu_i = 0$. Since for all $i \neq 2$ we have $u_2u_t \neq 0$, $u_t = u_2$ and consequently $u_1u_i = u_2$, for all $i \neq 2$.

(*v*). Suppose that $u_iu_j = x$, for all $i, j \neq 2$. By (iv), we have $u_1u_i = u_2$, for all $i \neq 2$. Hence $0 = u_1x = u_1(u_iu_j) = (u_1u_i)u_j = u_2u_j$, which is impossible and so $u_iu_j \neq x$. If, for all $i, j \neq 2$, we have $u_i^2 = 0$, then $0 = u_1u_i^2 = (u_1u_i)u_i = u_2u_i$, which is again impossible. Thus $u_iu_j \notin \{x, 0\}$, for all $i, j \neq 2$, and so there is t such that $u_iu_j = u_t$. If $t \neq 2$, then by (*iii*), $x = (u_2u_t) = u_2(u_iu_j) = (u_2u_i)u_j = xu_j = 0$, which is impossible. Therefore $u_iu_j = u_2$, for all $i, j \neq 2$. □

Lemma 3.12. *Suppose that* $\Gamma(S)$ *is a star graph with center* x *and end vertices* $U = \{u_1, u_2, \ldots, u_n\}$. *Also assum that* $u_1u_2 = x$ *and there is* $k \in \{3, 4, \ldots, n\}$ *such that* $u_1u_k \neq x$. *Then* u_i *is adjacent to* u_j *in* $AG(S)$, *for all* $i, j \in \{1, 2, \ldots, n\}$.

Proof. First, we show that u_i is adjacent to u_2 in $AG(S)$ for all $i \neq 2$. By Lemma 3.11 (*iii*), $u_2u_i = x$, for all $i \neq 2$ and $x^2 = 0$. Also for all i, we have $u_ix = 0$. Thus $\mathrm{ann}_S(x) = Z(S)$. Since $u_1u_i = u_2 \neq 0$ and $u_1u_2 = x \neq 0$, we have $u_1 \notin \mathrm{ann}_S(u_i) \cup \mathrm{ann}_S(u_2)$ which follows that $\mathrm{ann}_S(u_i) \cup \mathrm{ann}_S(u_2) \neq Z(S) = \mathrm{ann}_S(x) = \mathrm{ann}_S(u_i u_2)$. Therefore u_i is adjacent to u_2 in $AG(S)$, for all $i \neq 2$.

Now, we prove that u_i is adjacent to u_j in $AG(S)$, for all $i, j \neq 2$ and $i \neq j$. By items (ii), (iii) and (v) of Lemma 3.11, we have $u_2 u_i = x$ and $u_i u_j = u_2$, for all $i, j \neq 2$ and $u_2^2 = 0$. Thus, for all $i, j \neq 2$, we have $u_2 \notin \mathrm{ann}_S(u_i) \cup \mathrm{ann}_S(u_j)$ and $u_2 \in \mathrm{ann}_S(u_2) = \mathrm{ann}_S(u_i u_j)$. Therefore $\mathrm{ann}_S(u_i) \cup \mathrm{ann}_S(u_j) \neq \mathrm{ann}_S(u_2) = \mathrm{ann}_S(u_i u_j)$. Consequently u_i is adjacent to u_j in $AG(S)$, for all $i, j \neq 2$ and $i \neq j$. □

Now we have the following theorem.

Theorem 3.13. *Suppose that $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \ldots, u_n\}$. Also assume that $u_1 u_2 = x$ and there is $k \in \{3, 4, \ldots, n\}$ such that $u_1 u_k \neq x$. Then we have the following statements.*

- (i) *If $Z(S) \neq S$, then $AG(S) \cong K_{n+1}$.*
- (ii) *If $Z(S) = S$, then $AG(S) \cong K_1 \cup K_n$ with x is an isolated vertex.*

Proof. If $Z(S) \neq S$, then by [1, Theorem 3.1], we have $\Gamma(S) \leqslant AG(S)$ and if $Z(S) = S$, then by [1, Theorem 3.8], we have x is an isolated vertex in $AG(S)$. Now, by Lemma 3.12, the results hold. □

Example 3.14. Suppose that S is a commutative semigroup with zero and $Z^*(S) = \{w, x, y, z\}$ and $xy = xz = xw = 0$. Also assume that $yz = wy = x$, $wz = y$, $x^2 = y^2 = 0$ and $w^2 = z^2 = y$. Then $\Gamma(S)$ is a star graph $K_{1,3}$ with center x and end vertices $U = \{w, y, z\}$ and if $Z(S) \neq S$, then by Theorem 3.13, we have $AG(S) \cong K_4$ and if $Z(S) = S$, then by Theorem 3.13, we have $AG(S) \cong K_1 \cup K_3$ with x is an isolated vertex in $AG(S)$. The zero divisor graph and the annihilator graph are pictured in Figure 3.

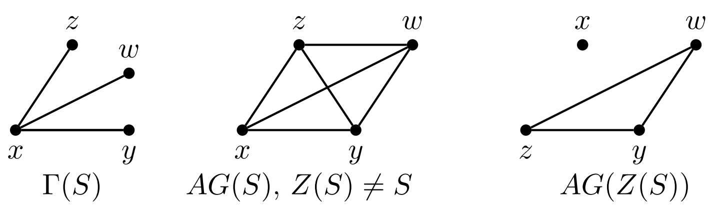


Figure 3

Example 3.15. Let $S = \{\bar{0}, \bar{1}, \bar{2}, \bar{4}, \bar{8}, \bar{10}\}$ be a semigroup with multiplicative operation modulo $\bar{16}$. Then $\Gamma(S)$ is a star graph with center $\bar{8}$ and end vertices $U = \{\bar{2}, \bar{4}, \bar{10}\}$. Also $\bar{2}.\bar{4} = \bar{4}.\bar{10} = \bar{8}$ and $\bar{2}.\bar{10} = \bar{4}$. Then $AG(S)$ is a star graph with center $\bar{8}$ and end vertices $U = \{\bar{2}, \bar{4}, \bar{10}\}$, and $AG(Z(S)) \cong K_1 \cup K_3$ with $\bar{8}$ is an isolated vertex.

The next corollary follows from Theorems 3.3 and 3.13.

Corollary 3.16. *Suppose that $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \dots, u_n\}$. Also assum that there are $i, j \in \{1, 2, \dots, n\}$ such that $u_i u_j = x$. Then one of the following statements holds.*

(*i*) *If $Z(S) \neq S$, then $AG(S) \cong K_{n+1}$.*

(*ii*) *If $Z(S) = S$, then $AG(S) \cong K_1 \cup K_n$ with x is an isolated vertex.*

The next corollary follows from Theorems 3.3, 3.8 and 3.13.

Corollary 3.17. *Suppose that $\Gamma(S)$ is a star graph. Then we have the following statements.*

(*i*) *If $Z(S) \neq S$, then $AG(S)$ is a star graph or a complete graph.*

(*ii*) *If $Z(S) = S$, then $AG(S)$ is an empty graph or a complete graph with an isolated vertex.*

(*iii*) *If $Z(S) = S$, then $AG(S)$ is a disconnected graph.*

In [2], we have showed that if $AG(S)$ is a star graph $K_{1,3}$ with center x, then $\Gamma(S)$ is a star graph $K_{1,3}$ with center x but the converse statement doesn't hold in general. In the following theorem, we provide some conditions under which the converse statement holds.

Theorem 3.18. *Suppose that S is a commutative semigroup and $Z(S) \neq S$. Then $AG(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \dots, u_n\}$ if and only if $\Gamma(S)$ is a star graph with center x, end vertices $U = \{u_1, u_2, \dots, u_n\}$ and $u_i u_j \neq x$, for all $i, j \in \{1, 2, \dots, n\}$.*

Proof. Suppose that $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \dots, u_n\}$ and $u_i u_j \neq x$, for all $i, j \in \{1, 2, \dots, n\}$. By Theorem 3.8, $AG(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \dots, u_n\}$.

Conversely, suppose that $Z(S) \neq S$ and $AG(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \dots, u_n\}$. Then $\Gamma(S) \leqslant AG(S)$ and if there are i and j that u_i is adjacent to u_j in $\Gamma(S)$, then u_i is adjacent to u_j in $AG(S)$, which is impossible. Since $\Gamma(S)$ is a connected graph, we have x is adjacent to u_i, for all $i \in \{1, 2, \dots, n\}$. Thus $\Gamma(S)$ is a star graph with center x and end vertices $U = \{u_1, u_2, \dots, u_n\}$. Now, if there are $i, j \in \{1, 2, \dots, n\}$ such that $u_i u_j = x$, then by Corollary 3.16, $AG(S) \cong K_{n+1}$, which is impossible. Therefore $u_i u_j \neq x$, for all $i, j \in \{1, 2, \dots, n\}$. □

In the following, we show that if $Z(S) = S$, then $AG(S)$ is never a star graph. First note that in [1, Theorem 4.3], it was proved that if $Z(S) = \{0, x, y, z\} = S$ and $xy = xz = 0$, then $AG(S) \cong K_2 \cup K_1$ with x is an isolated vertex if and only if $zy = x$ and we either have $y^2 \neq 0$ or $z^2 \neq 0$. Otherwise $AG(S) \cong 3K_1$. Therefore in this situation, we have $AG(S)$ is not a star graph when $Z(S) = S$.

Also, let S be a commutative semigroup with $Z(S) = S$. If $\Gamma(S)$ is not isomorphic to P_3, then by [2, Corollary 4.3], $AG(S)$ contains at least one isolated vertex, and thus $AG(S)$ is disconnected and it is not a star graph. Moreover, by [2, Lemma 4.4], $AG(S) \cong P_3$ with $x \sim w \sim z \sim y$ if and only if $\Gamma(S) \cong P_3$ with $w \sim x \sim y \sim z$. Therefore in this situation, we also have $AG(S)$ is not a star graph when $Z(S) = S$.

In the following theorem we prove that in general, if $Z(S) = S$, then $AG(S)$ is not a star graph.

Theorem 3.19. *Let $Z(S) = S$ and $n \geqslant 3$. Then $AG(S)$ is not a star graph $K_{1,n}$.*

Proof. Suppose that $Z(S) = S$ and $AG(S)$ is a star graph $K_{1,n}$ with center x and A is the set of all end vertices of $AG(S)$. If for all $y \in A$, we have $xy = 0$, then by [1, Theorem 3.8], x is an isolated vertex in $AG(S)$, which is impossible.

Now, suppose that there is $y \in A$ that $xy \neq 0$. Since x is adjacent to y in $AG(S)$, we have $xy \neq x$, $xy \neq y$ and so there is $z \in S = Z(S) \setminus \{x, y\}$ such that $xy = z$. Also $\mathrm{ann}_S(x) \cup \mathrm{ann}_S(y) \neq \mathrm{ann}_S(xy) = \mathrm{ann}_S(z)$. Therefore there is $w \in Z(S)$ that $w \notin \mathrm{ann}_S(x) \cup \mathrm{ann}_S(y)$, $w \in \mathrm{ann}_S(xy) = \mathrm{ann}_S(z)$ and one of the following two statements holds.

(*i*) If $w \in Z(S) \setminus \{x, y\}$, then $wx \neq 0$, $wy \neq 0$, $wz = 0$ and $xy = z \neq 0$.

(*ii*) If there is no $w \in Z(S) \setminus \{x, y\}$ such that $w \notin \mathrm{ann}_S(x) \cup \mathrm{ann}_S(y)$, then for all $w \in Z(S) \setminus \{x, y\}$ we have $w \in \mathrm{ann}_S(x) \cup \mathrm{ann}_S(y)$, and so $w \in \mathrm{ann}_S(x)$ or $w \in \mathrm{ann}_S(y)$. Then $wx = 0$ or $wy = 0$ and thus $wz = 0$. Therefore z is adjacent to all vertices in $Z(S) \setminus \{x, y\}$ in

$\Gamma(S)$, and so $\mathrm{ann}_S(z) \supseteq Z(S) \setminus \{x, y, z\}$. Also $w = y$ or $w = x$ and we have one of the following two statements.

(1) If $w = y$, then $xy = z \neq 0$, $y^2 \neq 0$ and $yz = y^2x = 0$.

(2) If $w = x$, then $xy = z \neq 0$, $x^2 \neq 0$ and $xz = x^2y = 0$.

If (i) holds, then since $wx \neq 0$ and $xy \neq 0$, we have $x \notin \mathrm{ann}_S(w) \cup \mathrm{ann}_S(y)$. On the other hand, since $x(wy) = w(xy) = wz = 0$, we have $x \in \mathrm{ann}_S(wy)$, and thus w is adjacent to y in $AG(S)$, which is impossible.

Now we show that $\mathrm{ann}_S(y) \subseteq \mathrm{ann}_S(x)$. Suppose that $u \in Z(S)\setminus\{x, y, z\}$ and $u \in \mathrm{ann}_S(y)$. Then $uy = 0$ and since u is not adjacent to y in $AG(S)$, we have $\mathrm{ann}_S(u)\cup\mathrm{ann}_S(y) = \mathrm{ann}_S(uy) = Z(S)$. Since $x \in Z(S) = \mathrm{ann}_S(uy) = \mathrm{ann}_S(u) \cup \mathrm{ann}_S(y)$ and $xy \neq 0$, we have $x \in \mathrm{ann}_S(u)$ and so $xu = 0$. Therefore $u \in \mathrm{ann}_S(x)$. Consequently $\mathrm{ann}_S(y) \subseteq \mathrm{ann}_S(x)$.

Now, assume that (1) is true. Then $yz = 0$ and so $z \in \mathrm{ann}_S(y) \subseteq \mathrm{ann}_S(x)$. Thus $xz = 0$. Therefore $x \in \mathrm{ann}_S(z)$ and so $\mathrm{ann}_S(z) \supseteq Z(S)\setminus\{z\}$. By [1, Theorem 3.8], z is an isolated vertex in $AG(S)$, which is impossible.

Now let (2) be true. If $x^2 = x$, then $z = xy = x^2y = x(xy) = xz = 0$, which is impossible. If $x^2 = y$, then $z = xy = xx^2 = x^3$, and so $yz = x^3y = (x^2y)x = 0x = 0$. Therefore $y \in \mathrm{ann}_S(z)$ and thus $\mathrm{ann}_S(z) \supseteq Z(S) \setminus \{z\}$. By [1, Theorem 3.8], z is an isolated vertex in $AG(S)$, which is impossible. If $x^2 \notin \{x, y\}$, then x^2 is not adjacent to y in $AG(S)$, and so $\mathrm{ann}_S(x^2) \cup \mathrm{ann}_S(y) = \mathrm{ann}_S(x^2y) = Z(S)$. Since $\mathrm{ann}_S(y) \subseteq \mathrm{ann}_S(x) \subseteq \mathrm{ann}_S(x^2)$, we have $\mathrm{ann}_S(x^2) = Z(S)$ and by [1, Theorem 3.8], x^2 is an isolated vertex in $AG(S)$, which is impossible. □

We end this section with a corollary which follows from Theorems 3.18 and 3.19.

Corollary 3.20. *Suppose that $AG(S)$ is a star graph with center x. Then $Z(S) \neq S$ and $\Gamma(S)$ is a star graph with center x, and the product of all end vertices of $\Gamma(S)$ is not equal to the center vertex.*

Acknowledgments. The authors are very grateful to the referees for their comments and suggestions.

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Received May 20, 2025

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