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Green's relations on a semigroup of transformations with restricted partial range

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Abstract. Let $T(X)$ denote the semigroup of all full transformations on a nonempty set X. For every pair of nonempty subsets Y and Z of X with $Z \subseteq Y$, we denote by $T(X, Y, Z)$ the subsemigroup of $T(X)$ consisting of all transformations $\alpha \in T(X)$ such that $Y\alpha \subseteq Z$. We characterize Green's relations on the subsemigroup $T(X, Y, Z)$.

1. Introduction and preliminaries

In 1951, Green [4] introduced equivalence relations on a semigroup S, namely $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{J}$ by the following rules: $(a, b) \in \mathcal{R}$ if $aS^1 = bS^1$, $(a, b) \in \mathcal{L}$ if $S^1a = S^1b$, and $(a, b) \in \mathcal{J}$ if $S^1aS^1 = S^1bS^1$, where S^1 is the monoid obtained from S by adding an identity if S contains no identity, otherwise, $S^1 = S$. He further defined the equivalence relation $\mathcal{H}$ on S as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and the equivalence relation $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$. The equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$, and $\mathcal{D}$ on S are known as Green's relations after J. A. Green who introduced them. It is well-known that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ (cf.[6, Proposition 2.1.3]), and consequently $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ (cf. [6, Corollary 1.5.12]). There have been many research works investigating Green's relations on various semigroups of transformations. For a broader treatment of related concepts, see [3, 1].

Let X be a nonempty set. Denote by $T(X)$ the semigroup of all full transformations on X. Semigroup $T(X)$ and its subsemigroups are crucial in semigroup theory, since every semigroup can be embedded in $T(Z)$ for some appropriate set Z (cf. [6, Theorem 1.1.2]). Miller and Doss [2] characterized Green's relations on $T(X)$. Green's relations on various subsemigroups of $T(X)$ are also characterized. In particular, for a nonempty subset Y of X, Magill [9] in 1966 introduced the subsemigroup $\overline{T}(X, Y) := \{\alpha \in$

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$T(X)\colon Y\alpha \subseteq Y\}$ of $T(X)$; Symons [16] in 1975 introduced the subsemigroup $T(X,Y) := \{\alpha \in T(X)\colon X\alpha \subseteq Y\}$ of $T(X)$. Honyam and Sanwong [5] characterized Green's relations on $\overline{T}(X,Y)$. Sanwong and Sommanee [11] characterized Green's relations on $T(X,Y)$. Several other properties of both the semigroups $\overline{T}(X,Y)$ and $T(X,Y)$ have also been investigated (see, the respective lists [5, 9, 10, 13, 15] and [10, 11, 14, 16, 17] for some references).

For every pair of nonempty subsets Y and Z of X with $Z \subseteq Y$, consider the subsemigroup $T(X,Y,Z)$ of $T(X)$ consisting of all transformations $\alpha \in T(X)$ such that $Y\alpha \subseteq Z$. Using symbols,

$$T(X,Y,Z) := \{\alpha \in T(X)\colon Y\alpha \subseteq Z\}.$$

Jin [7] first studied the semigroup $T(X,Y,Z)$, and called it the *semigroup of transformations with restricted partial range on X*. Semigroup $T(X,Y,Z)$ may be regarded as a generalization of three known semigroups, namely $T(X)$, $T(X,Y)$, and $\overline{T}(X,Y)$ since

- $T(X,Y,Z) = T(X)$ if $Z = Y = X$,
- $T(X,Y,Z) = T(X,Z)$ if $Y = X$,
- $T(X,Y,Z) = \overline{T}(X,Y)$ if $Z = Y$.

Jin [7] calculated the rank of the semigroup $T(X,Y,Z)$ when X is finite and $1 \leq |Z| < |Y| < |X|$. Jin and You [8] characterized regular elements in $T(X,Y,Z)$, determined when $T(X,Y,Z)$ is regular, and investigated the abundance of $T(X,Y,Z)$. Jin and You [8] calculated the number of elements, regular elements, and idempotent elements in $T(X,Y,Z)$ when X is finite.

In this paper, we characterize Green's relations on $T(X,Y,Z)$. Before that, we introduce some useful concepts and notation, which are indispensable for this paper. Let X be a nonempty set. The cardinality of X is denoted by $|X|$. The identity mapping on X is denoted by id_X. A *partition* of X is a collection of pairwise disjoint nonempty subsets, called *blocks*, whose union is X. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of X, and let $\mathcal{A}$ and $\mathcal{B}$ be subcollections of blocks in $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then $\mathcal{A}$ *refines* $\mathcal{B}$, denoted by $\mathcal{A} \preceq \mathcal{B}$, if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$. Moreover, we write $\mathcal{A} = \mathcal{B}$ if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{A}$. For any sets A and B, let $A \setminus B := \{x \in A\colon x \notin B\}$. We compose mappings from left to right and denote their composition by juxtaposition. Let $f\colon X \to Y$ be a mapping. We write xf to denote the image of an element x of X under f. For a subset A of X, let $Af := \{af\colon a \in A\}$. For a subset B of Y, let

$Bf^{-1} := \{x \in X : xf \in B\}$. Moreover, if $B = \{b\}$ then we write bf^{-1} to denote the set $\{b\}f^{-1}$. We also refer the readers to [6] for any concepts and notation not defined here.

2. Main results

Assume throughout this section that Y and Z are nonempty subsets of X such that $Z \subseteq Y$. In general, the semigroup $T(X, Y, Z)$ does not contain the identity transformation id_X, and so $T(X, Y, Z)^1 \neq T(X, Y, Z)$.

Firstly we characterize Green's relations on the semigroup $T(X, Y, Z)$. We begin by recalling the following lemma from [5], which characterizes the relation $\mathcal{L}$ on the semigroup $\overline{T}(X, Y)$.

Lemma 2.1. [5, Lemma 2] *Let $\alpha, \beta \in \overline{T}(X, Y)$. Then $\alpha = \gamma\beta$ for some $\gamma \in \overline{T}(X, Y)$ if and only if $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$. Consequently, $(\alpha, \beta) \in \mathcal{L}$ in $\overline{T}(X, Y)$ if and only if $X\alpha = X\beta$ and $Y\alpha = Y\beta$.*

Lemma 2.2. *Let $\alpha, \beta \in T(X, Y, Z)$. Then $\alpha = \gamma\beta$ for some $\gamma \in T(X, Y, Z)^1$ if and only if $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Z\beta$.*

Proof. Assume that $\alpha = \gamma\beta$ for some $\gamma \in T(X, Y, Z)^1$. Then $X\alpha \subseteq X\beta$ by Lemma 2.1, since $T(X, Y, Z)^1 \subseteq \overline{T}(X, Y)$. Now since $\alpha = \gamma\beta$ and $Y\gamma \subseteq Z$, we obtain $Y\alpha = (Y\gamma)\beta \subseteq Z\beta$.

Conversely, assume that the given conditions hold. Since $Y\alpha \subseteq Z\beta$, for each $x \in Y$ we fix $z_x \in Z$ such that $x\alpha = z_x\beta$. Since $X\alpha \subseteq X\beta$, for each $x \in X$ we fix $x' \in X$ such that $x\alpha = x'\beta$. Define a mapping $\gamma : X \to X$ by

$$x\gamma = \begin{cases} z_x & \text{if } x \in Y, \\ x' & \text{if } x \in X \setminus Y. \end{cases}$$

Observe that $Y\gamma \subseteq Z$, and so $\gamma \in T(X, Y, Z)$. Finally, to prove that $\alpha = \gamma\beta$, let $x \in X$. If $x \in Y$, then $x\gamma = z_x$. Therefore $x(\gamma\beta) = (x\gamma)\beta = z_x\beta = x\alpha$. On the other hand, if $x \in X \setminus Y$ then $x\gamma = x'$. Therefore $x(\gamma\beta) = (x\gamma)\beta = x'\beta = x\alpha$. Hence $\alpha = \gamma\beta$. □

Theorem 2.3. *Let $\alpha, \beta \in T(X, Y, Z)$. Then $(\alpha, \beta) \in \mathcal{L}$ in $T(X, Y, Z)$ if and only if $X\alpha = X\beta$, $Y\alpha \subseteq Z\beta$, and $Y\beta \subseteq Z\alpha$.*

Proof. It follows directly from Lemma 2.2. □

The following corollary is immediate from Theorem 2.3, since $Z \subseteq Y$.

Corollary 2.4. *If* $(\alpha, \beta) \in \mathcal{L}$ *in* $T(X, Y, Z)$*, then* $Y\alpha = Z\alpha = Z\beta = Y\beta$*.*

For every $\alpha \in T(X)$ and $A \subseteq X$ such that $X\alpha \cap A \neq \varnothing$, let

$$\pi_\alpha(A) := \{y\alpha^{-1} \colon y \in X\alpha \cap A\}.$$

Moreover, we write the notation π_α to denote the set $\pi_\alpha(X)$. Note that π_α is a partition of X induced by the transformation α.

Honyam and Sanwong [5] provided the following lemma, which characterizes the relation $\mathcal{R}$ on the semigroup $\overline{T}(X, Y)$.

Lemma 2.5. [5, Lemma 3] *Let* $\alpha, \beta \in \overline{T}(X, Y)$*. Then* $\alpha = \beta\gamma$ *for some* $\gamma \in \overline{T}(X, Y)$ *if and only if* $\pi_\beta \preceq \pi_\alpha$ *and* $\pi_\beta(Y) \preceq \pi_\alpha(Y)$*. Consequently,* $(\alpha, \beta) \in \mathcal{R}$ *in* $\overline{T}(X, Y)$ *if and only if* $\pi_\beta = \pi_\alpha$ *and* $\pi_\beta(Y) = \pi_\alpha(Y)$*.*

The following lemma will allow us to characterize the relation $\mathcal{R}$ on the semigroup $T(X, Y, Z)$.

Lemma 2.6. *Let* $\alpha, \beta \in T(X, Y, Z)$*. Then* $\alpha = \beta\gamma$ *for some* $\gamma \in T(X, Y, Z)^1$ *if and only if* $\pi_\beta \preceq \pi_\alpha$ *and* $\pi_\beta(Y) \preceq \pi_\alpha(Z)$*.*

Proof. Assume that $\alpha = \beta\gamma$ for some $\gamma \in T(X, Y, Z)^1$. Then $\pi_\beta \preceq \pi_\alpha$ by Lemma 2.5, since $T(X, Y, Z)^1 \subseteq \overline{T}(X, Y)$. Next, to prove that $\pi_\beta(Y) \preceq \pi_\alpha(Z)$, let $A \in \pi_\beta(Y)$. Then $A = y\beta^{-1}$ for some $y \in X\beta \cap Y$. Since $\alpha = \beta\gamma$, we obtain $A\alpha = (A\beta)\gamma = y\gamma$. Therefore $A \subseteq (y\gamma)\alpha^{-1}$. It remains to show that $y\gamma \in X\alpha \cap Z$. Since $\beta\gamma = \alpha$ and $Y\gamma \subseteq Z$, we obtain $(X\beta \cap Y)\gamma \subseteq (X\beta)\gamma \cap Y\gamma \subseteq X\alpha \cap Z$. Since $y \in X\beta \cap Y$, it follows that $y\gamma \in X\alpha \cap Z$. Hence $\pi_\beta(Y) \preceq \pi_\alpha(Z)$.

Conversely, assume that the given conditions hold. For each $x \in X\beta$, we fix $z_x \in x\beta^{-1}$. Define a mapping $\gamma \colon X \to X$ by

$$x\gamma = \begin{cases} z_x\alpha & \text{if } x \in X\beta, \\ x\beta & \text{if } x \in X \setminus X\beta. \end{cases}$$

Observe that γ is well-defined, since $\pi_\beta \preceq \pi_\alpha$. It is clear that $\beta\gamma = \alpha$, since $x(\beta\gamma) = (x\beta)\gamma = x\alpha$ for all $x \in X$. It is also easy to verify that $\gamma \in T(X, Z) \subseteq T(X, Y, Z)$. □

Theorem 2.7. *Let* $\alpha, \beta \in T(X, Y, Z)$*. Then* $(\alpha, \beta) \in \mathcal{R}$ *in* $T(X, Y, Z)$ *if and only if* $\pi_\alpha = \pi_\beta$*,* $\pi_\alpha(Y) \preceq \pi_\beta(Z)$*, and* $\pi_\beta(Y) \preceq \pi_\alpha(Z)$*.*

Proof. It follows directly from Lemma 2.6. □

The following corollary is immediate from Theorem 2.7, since $Z \subseteq Y$.

Corollary 2.8. *If* $(\alpha, \beta) \in \mathcal{R}$ *in* $T(X, Y, Z)$*, then* $\pi_\alpha(Y) = \pi_\alpha(Z)$ *and* $\pi_\beta(Y) = \pi_\beta(Z)$*.*

Corollary 2.9. *If* $(\alpha, \beta) \in \mathcal{R}$ *in* $T(X, Y, Z)$*, then*

(i) $X\alpha \subseteq Z \cup (X \setminus Y)$,

(ii) $X\beta \subseteq Z \cup (X \setminus Y)$,

(iii) $X\alpha \setminus Z = X\alpha \setminus Y$,

(iv) $X\beta \setminus Z = X\beta \setminus Y$.

Proof. Assume that $(\alpha, \beta) \in \mathcal{R}$ in $T(X, Y, Z)$.

(i). Suppose to the contrary that there exists $x \in X$ such that $x\alpha \in Y \setminus Z$. Then $x\alpha \in X\alpha \cap Y$, and so $(x\alpha)\alpha^{-1} \in \pi_\alpha(Y)$. Since $\pi_\alpha(Y) = \pi_\alpha(Z)$ by Corollary 2.8, we obtain $(x\alpha)\alpha^{-1} \in \pi_\alpha(Z)$. Therefore $x\alpha \in Z$, a contradiction.

(ii). The proof proceeds similarly to that of (i), with β substituted for α throughout the argument.

(iii). By (i), we see that $X\alpha \cap (Y \setminus Z) = \varnothing$. Now consider $X\alpha \setminus Y = X\alpha \setminus \big(Z \cup (Y \setminus Z)\big) = (X\alpha \setminus Z) \cap \big(X\alpha \setminus (Y \setminus Z)\big) = (X\alpha \setminus Z) \cap X\alpha = X\alpha \setminus Z$.

(iv). By (ii), we see that $X\beta \cap (Y \setminus Z) = \varnothing$. The remaining proof proceeds similarly to that of (iii), with β substituted for α throughout the argument. □

The following theorem characterizes the relation $\mathcal{D}$ on the semigroup $T(X, Y, Z)$.

Theorem 2.10. *Let* $\alpha, \beta \in T(X, Y, Z)$*. Then* $(\alpha, \beta) \in \mathcal{D}$ *in* $T(X, Y, Z)$ *if and only if* $(\alpha, \beta) \in \mathcal{D}$ *in* $\overline{T}(X, Y)$*.*

Proof. Assume that $(\alpha, \beta) \in \mathcal{D}$ in $T(X, Y, Z)$. Then $(\alpha, \beta) \in \mathcal{D}$ in $\overline{T}(X, Y)$, since $T(X, Y, Z)^1 \subseteq \overline{T}(X, Y)$.

Conversely, assume that $(\alpha, \beta) \in \mathcal{D}$ in $\overline{T}(X, Y)$. Then, since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, there exists $\gamma \in \overline{T}(X, Y)$ such that $(\alpha, \gamma) \in \mathcal{L}$ and $(\gamma, \beta) \in \mathcal{R}$ in $\overline{T}(X, Y)$. To show that $(\alpha, \beta) \in \mathcal{D}$ in $T(X, Y, Z)$, it suffices to prove that $\gamma \in T(X, Y, Z)^1$. Observe that $\alpha \in \overline{T}(X, Y)$, since $T(X, Y, Z)^1 \subseteq \overline{T}(X, Y)$. Thus, by Lemma 2.1 we obtain $Y\alpha = Y\gamma$. Therefore $Y\gamma \subseteq Z$, since $Y\alpha \subseteq Z$. Hence $\gamma \in T(X, Y, Z)^1$. □

Honyam and Sanwong [5] provided the following theorem, which characterizes the relation $\mathcal{J}$ on the semigroup $\overline{T}(X,Y)$.

Theorem 2.11. [5, Theorem 5] *Let $\alpha,\beta \in \overline{T}(X,Y)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda,\mu \in \overline{T}(X,Y)$ if and only if $|X\alpha| \leq |X\beta|$, $|Y\alpha| \leq |Y\beta|$, and $|X\alpha \setminus Y| \leq |X\beta \setminus Y|$. Consequently, $(\alpha,\beta) \in \mathcal{J}$ in $\overline{T}(X,Y)$ if and only if $|X\alpha| = |X\beta|$, $|Y\alpha| = |Y\beta|$, and $|X\alpha \setminus Y| = |X\beta \setminus Y|$.*

The following lemma will allow us to characterize the relation $\mathcal{J}$ on the semigroup $T(X,Y,Z)$.

Lemma 2.12. *Let $\alpha,\beta \in T(X,Y,Z)$. Then $\alpha = \gamma\beta\mu$ for some $\gamma,\mu \in T(X,Y,Z)^1$ if and only if there exists $\varphi \in \overline{T}(X,Z)$ such that*

(i) *$\alpha = \varphi\beta\rho$ for some $\rho \in \overline{T}(X,Z)$,*

(ii) *$\pi_{\varphi\beta}(Y) \preceq \pi_\alpha(Z)$,*

(iii) *$Y\alpha \subseteq Z(\beta\vartheta)$ for some $\vartheta \in T(X,Y,Z)^1$.*

Proof. Assume that $\alpha = \gamma\beta\mu$ for some $\gamma,\mu \in T(X,Y,Z)^1$.

(i). It is obvious, since $T(X,Y,Z)^1 \subseteq \overline{T}(X,Z)$.

(ii). Since $\alpha = (\gamma\beta)\mu$, we obtain $\pi_{\gamma\beta}(Y) \preceq \pi_\alpha(Z)$ by Lemma 2.6.

(iii). Since $\alpha = \gamma(\beta\mu)$, we obtain $Y\alpha \subseteq Z(\beta\mu)$ by Lemma 2.2.

Conversely, assume that the given conditions hold. For each $y \in Y$, by (iii) we fix $z_y \in Z$ such that $y\alpha = z_y(\beta\vartheta)$. Define a mapping $\phi\colon X \to X$ by

$$x\phi = \begin{cases} x\varphi & \text{if } x \in Z \cup (X \setminus Y), \\ z_x & \text{if } x \in Y \setminus Z. \end{cases}$$

Observe that $\phi \in T(X,Y,Z)$, since $z_x \in Z$ and $\varphi \in \overline{T}(X,Z)$. Now for each $y \in X(\varphi\beta) \cap Y$, by (ii) we fix $a_y \in X\alpha \cap Z$ such that $y(\varphi\beta)^{-1} \subseteq a_y\alpha^{-1}$. Also fix $z' \in Z$. Define a mapping $\eta\colon X \to X$ by

$$x\eta = \begin{cases} a_x & \text{if } x \in X(\varphi\beta) \cap Y, \\ z' & \text{otherwise.} \end{cases}$$

Observe that $\eta \in T(X,Y,Z)$, since $a_x, z' \in Z$. Now we choose $z_0 \in Z$, and define a mapping $\sigma\colon X \to X$ by

$$x\sigma = \begin{cases} z_0 & \text{if } x \in X \setminus X(\phi\beta), \\ x\vartheta & \text{if } x \in X(\phi\beta) \cap (Y \setminus Z), \\ x\eta & \text{if } x \in X(\phi\beta) \cap (X \setminus Y), \\ x\rho & \text{if } x \in X(\phi\beta) \cap Z. \end{cases}$$

To prove that $\sigma \in T(X,Y,Z)$, let $x \in Y$. Then either $x \in Y \setminus X(\phi\beta)$ or $x \in Y \cap X(\phi\beta)$. If $x \in Y \setminus X(\phi\beta)$, then $x\sigma = z_0 \in Z$. On the other hand, let $x \in Y \cap X(\phi\beta)$. Then either $x \in Z \cap X(\phi\beta)$ or $x \in (Y \setminus Z) \cap X(\phi\beta)$. If $x \in Z \cap X(\phi\beta)$, then $x\sigma = x\rho$ by the definition of σ. Note that $Z\rho \subseteq Z$, and so $x\sigma \in Z$. So, assume that $x \in (Y \setminus Z) \cap X(\phi\beta)$. Then $x\sigma = x\vartheta$ by the definition of σ. Since $x \in Y$ and $Y\vartheta \subseteq Z$, it follows that $x\sigma = x\vartheta \in Z$. Hence $\sigma \in T(X,Y,Z)$. Finally, we prove that $\alpha = \phi\beta\sigma$. Let $x \in X$. We then proceed by considering three different cases:
Case 1: $x \in Z$. Then $x\phi = x\varphi$ by the definition of ϕ. Therefore $x(\phi\beta\sigma) = (x(\phi\beta))\,\sigma = (x(\phi\beta))\,\rho = x\phi(\beta\rho) = x\varphi(\beta\rho) = x(\varphi\beta\rho) = x\alpha$.
Case 2: $x \in Y \setminus Z$. Then $x\phi = z_x$ by the definition of ϕ. Therefore $x(\phi\beta\sigma) = (x(\phi\beta))\,\sigma = (x(\phi\beta))\,\vartheta = x\phi(\beta\vartheta) = z_x(\beta\vartheta) = x\alpha$.
Case 3: $x \in X \setminus Y$. Then $x\phi = x\varphi$ by the definition of ϕ. Therefore $x(\phi\beta\sigma) = (x(\phi\beta))\,\sigma = (x(\phi\beta))\,\eta = x\phi(\beta\eta) = x\varphi(\beta\eta) = (x(\varphi\beta))\,\eta = a_x = x\alpha$.

In each case, we have $x(\phi\beta\sigma) = x\alpha$. Hence $\phi\beta\sigma = \alpha$. □

Theorem 2.13. *Let $\alpha, \beta \in T(X,Y,Z)$. Then $(\alpha,\beta) \in \mathcal{J}$ in $T(X,Y,Z)$ if and only if there exist $\varphi, \eta \in \overline{T}(X,Z)$ such that*

(i) *$\alpha = \varphi\beta\rho$ and $\beta = \eta\alpha\sigma$ for some $\rho, \sigma \in \overline{T}(X,Z)$,*

(ii) *$\pi_{\varphi\beta}(Y) \preceq \pi_\alpha(Z)$ and $\pi_{\eta\alpha}(Y) \preceq \pi_\beta(Z)$,*

(iii) *$Y\alpha \subseteq Z(\beta\vartheta)$ and $Y\beta \subseteq Z(\alpha\gamma)$ for some $\vartheta, \gamma \in T(X,Y,Z)^1$.*

Proof. It follows directly from Lemma 2.12. □

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